Non-Hausdorff groupoids, proper actions and K-theory

Jean-Louis Tu

Abstract. Let G be a (not necessarily Hausdorff) locally compact groupoid. We introduce a notion of properness for G, which is invariant under Morita-equivalence. We show that any generalized morphism between two locally compact groupoids which satisfies some properness conditions induces a C^*-correspondence from $C_r^*(G_2)$ to $C_r^*(G_1)$, and thus two Morita equivalent groupoids have Morita-equivalent C^*-algebras.

2000 Mathematics Subject Classification: 22A22 (Primary); 46L05, 46L80, 54D35 (Secondary).
Keywords and Phrases: groupoid, C^*-algebra, K-theory.

Introduction

Very often, groupoids that appear in geometry, such as holonomy groupoids of foliations, groupoids of inverse semigroups [15, 6] and the indicial algebra of a manifold with corners [10] are not Hausdorff. It is thus necessary to extend various basic notions to this broader setting, such as proper action and Morita equivalence. We also show that a generalized morphism from G_2 to G_1 satisfying certain properness conditions induces an element of $KK(C_r^*(G_2), C_r^*(G_1))$.

In Section 2, we introduce the notion of proper groupoids and show that it is invariant under Morita-equivalence.
Section 3 is a technical part of the paper in which from every locally compact topological space X is canonically constructed a locally compact Hausdorff space $\mathcal{H}X$ in which X is (not continuously) embedded. When G is a groupoid (locally compact, with Haar system, such that $G^{(0)}$ is Hausdorff), the closure X' of $G^{(0)}$ in $\mathcal{H}G$ is endowed with a continuous action of G and plays an important technical rôle.
In Section 4 we review basic properties of locally compact groupoids with Haar system and technical tools that are used later.
In Section 5 we construct, using tools of Section 3, a canonical $C_r^*(G)$-Hilbert module $\mathcal{E}(G)$ for every (locally compact...) proper groupoid G. If $G^{(0)}/G$ is compact, then there exists a projection $p \in C_r^*(G)$ such that $\mathcal{E}(G)$ is isomorphic to $pC_r^*(G)$. The projection p is given by $p(g) = (c(s(g))c(r(g)))^{1/2}$, where $c\colon G^{(0)} \to \mathbb{R}_+$ is a "cutoff" function (Section 6). Contrary to the Hausdorff

case, the function c is not continuous, but it is the restriction to $G^{(0)}$ of a continuous map $X' \to \mathbb{R}_+$ (see above for the definition of X').
In Section 7, we examine the question of naturality $G \mapsto C^*_r(G)$. Recall that if $f\colon X \to Y$ is a continuous map between two locally compact spaces, then f induces a map from $C_0(Y)$ to $C_0(X)$ if and only if f is proper. When G_1 and G_2 are groups, a morphism $f\colon G_1 \to G_2$ does not induce a map $C^*_r(G_2) \to C^*_r(G_1)$ (when $G_1 \subset G_2$ is an inclusion of discrete groups there is a map in the other direction). When $f\colon G_1 \to G_2$ is a groupoid morphism, we cannot expect to get more than a C^*-correspondence from $C^*_r(G_2)$ to $C^*_r(G_1)$ when f satisfies certain properness assumptions: this was done in the Hausdorff situation by Macho-Stadler and O'Uchi ([11, Theorem 2.1], see also [7, 13, 17]), but the formulation of their theorem is somewhat complicated. In this paper, as a corollary of Theorem 7.8, we get that (in the Hausdorff situation), if the restriction of f to $(G_1)_K^K$ is proper for each compact set $K \subset (G_1)^{(0)}$ then f induces a correspondence $\mathcal{E}_f$ from $C^*_r(G_2)$ to $C^*_r(G_1)$. In fact we construct a C^*-correspondence out of any groupoid generalized morphism ([5, 9]) which satisfies some properness conditions. As a corollary, if G_1 and G_2 are Morita equivalent then $C^*_r(G_1)$ and $C^*_r(G_2)$ are Morita-equivalent C^*-algebras.

Finally, let us add that our original motivation was to extend Baum, Connes and Higson's construction of the assembly map μ to non-Hausdorff groupoids; however, we couldn't prove μ to be an isomorphism in any non-trivial case.

1. Preliminaries

1.1. Groupoids. Throughout, we will assume that the reader is familiar with basic definitions about groupoids (see [16, 15]). If G is a groupoid, we denote by $G^{(0)}$ its set of units and by $r\colon G \to G^{(0)}$ and $s\colon G \to G^{(0)}$ its range and source maps respectively. We will use notations such as $G_x = s^{-1}(x)$, $G^y = r^{-1}(y)$, $G_x^y = G_x \cap G^y$. Recall that a topological groupoid is said to be *étale* if r (and s) are local homeomorphisms.

For all sets X, Y, T and all maps $f\colon X \to T$ and $g\colon Y \to T$, we denote by $X \times_{f,g} Y$, or by $X \times_T Y$ if there is no ambiguity, the set $\{(x,y) \in X \times Y |\ f(x) = g(y)\}$.

Recall that a (right) action of G on a set Z is given by

(a) a ("momentum") map $p\colon Z \to G^{(0)}$;
(b) a map $Z \times_{p,r} G \to Z$, denoted by $(z,g) \mapsto zg$

with the following properties:

(i) $p(zg) = s(g)$ for all $(z,g) \in Z \times_{p,r} G$;
(ii) $z(gh) = (zg)h$ whenever $p(z) = r(g)$ and $s(g) = r(h)$;
(iii) $zp(z) = z$ for all $z \in Z$.

Then the crossed-product $Z \rtimes G$ is the subgroupoid of $(Z \times Z) \times G$ consisting of elements (z, z', g) such that $z' = zg$. Since the map $Z \rtimes G \to Z \times G$ given by $(z, z', g) \mapsto (z, g)$ is injective, the groupoid $Z \rtimes G$ can also be considered as a subspace of $Z \times G$, and this is what we will do most of the time.

1.2. LOCALLY COMPACT SPACES. A topological space X is said to be quasi-compact if every open cover of X admits a finite sub-cover. A space is compact if it is quasi-compact and Hausdorff. Let us recall a few basic facts about locally compact spaces.

DEFINITION 1.1. *A topological space X is said to be locally compact if every point $x \in X$ has a compact neighborhood.*

In particular, X is locally Hausdorff, thus every singleton subset of X is closed. Moreover, the diagonal in $X \times X$ is locally closed.

PROPOSITION 1.2. *Let X be a locally compact space. Then every locally closed subspace of X is locally compact.*

Recall that $A \subset X$ is locally closed if for every $a \in A$, there exists a neighborhood V of a in X such that $V \cap A$ is closed in V. Then A is locally closed if and only if it is of the form $U \cap F$, with U open and F closed.

PROPOSITION 1.3. *Let X be a locally compact space. The following are equivalent:*

(i) *there exists a sequence (K_n) of compact subspaces such that $X = \cup_{n\in\mathbb{N}} K_n$;*
(ii) *there exists a sequence (K_n) of quasi-compact subspaces such that $X = \cup_{n\in\mathbb{N}} K_n$;*
(iii) *there exists a sequence (K_n) of quasi-compact subspaces such that $X = \cup_{n\in\mathbb{N}} K_n$ and $K_n \subset \mathring{K}_{n+1}$ for all $n \in \mathbb{N}$.*

Such a space will be called σ-compact.

Proof. (i) $\Longrightarrow$ (ii) is obvious. The implications (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (i) follow easily from the fact that for every quasi-compact subspace K, there exists a finite family $(K_i)_{i\in I}$ of compact sets such that $K \subset \cup_{i\in I} \mathring{K}_i$. $\square$

1.3. PROPER MAPS.

PROPOSITION 1.4. [2, Théorème I.10.2.1] *Let X and Y be two topological spaces, and $f\colon X \to Y$ a continuous map. The following are equivalent:*

(i) *For every topological space Z, $f \times \mathrm{Id}_Z\colon X \times Z \to Y \times Z$ is closed;*
(ii) *f is closed and for every $y \in Y$, $f^{-1}(y)$ is quasi-compact.*

A map which satisfies the equivalent properties of Proposition 1.4 is said to be *proper*.

PROPOSITION 1.5. [2, Proposition I.10.2.6] *Let X and Y be two topological spaces and let $f\colon X \to Y$ be a proper map. Then for every quasi-compact subspace K of Y, $f^{-1}(K)$ is quasi-compact.*

PROPOSITION 1.6. *Let X and Y be two topological spaces and let $f\colon X \to Y$ be a continuous map. Suppose Y is locally compact, then the following are equivalent:*

(i) *f is proper;*

(ii) *for every quasi-compact subspace K of Y, $f^{-1}(K)$ is quasi-compact;*
(iii) *for every compact subspace K of Y, $f^{-1}(K)$ is quasi-compact;*
(iv) *for every $y \in Y$, there exists a compact neighborhood K_y of y such that $f^{-1}(K_y)$ is quasi-compact.*

Proof. (i) $\Longrightarrow$ (ii) follows from Proposition 1.5. (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) are obvious. Let us show (iv) $\Longrightarrow$ (i).
Since $f^{-1}(y)$ is closed, it is clear that $f^{-1}(y)$ is quasi-compact for all $y \in Y$. It remains to prove that for every closed subspace $F \subset X$, $f(F)$ is closed. Let $y \in \overline{f(F)}$. Let $A = f^{-1}(K_y)$. Then $A \cap F$ is quasi-compact, so $f(A \cap F)$ is quasi-compact. As $f(A \cap F) \subset K_y$, it is closed in K_y, i.e. $K_y \cap \overline{f(A \cap F)} = K_y \cap f(A \cap F)$. We thus have $y \in K_y \cap \overline{f(A \cap F)} = K_y \cap f(A \cap F) \subset f(F)$. It follows that $f(F)$ is closed. □

2. Proper groupoids and proper actions

2.1. Locally compact groupoids.

DEFINITION 2.1. *A topological groupoid G is said to be locally compact (resp. σ-compact) if it is locally compact (resp. σ-compact) as a topological space.*

REMARK 2.2. *The definition of a locally compact groupoid in* [15] *corresponds to our definition of a locally compact, σ-compact groupoid with Haar system whose unit space is Hausdorff, thanks to Propositions 2.5 and 2.8.*

EXAMPLE 2.3. *Let Γ be a discrete group, H a closed normal subgroup and let G be the bundle of groups over $[0,1]$ such that $G_0 = \Gamma$ and $G_t = \Gamma/H$ for all $t > 0$. We endow G with the quotient topology of $([0,1] \times \Gamma) \,/\, ((0,1] \times H)$. Then G is a non-Hausdorff locally compact groupoid such that $(t, \bar{\gamma})$ converges to $(0, \gamma h)$ as $t \to 0$, for all $\gamma \in \Gamma$ and $h \in H$.*

EXAMPLE 2.4. *Let Γ be a discrete group acting on a locally compact Hausdorff space X, and let $G = (X \times \Gamma)/\sim$, where (x, γ) and (x, γ') are identified if their germs are equal, i.e. there exists a neighborhood V of x such that $y\gamma = y\gamma'$ for all $y \in V$. Then G is locally compact, since the open sets $V_\gamma = \{[(x,\gamma)] |\, x \in X\}$ are homeomorphic to X and cover G.*
Suppose that X is a manifold, M is a manifold such that $\pi_1(M) = \Gamma$, $\tilde{M}$ is the universal cover of M and $V = (X \times \tilde{M})/\Gamma$, then V is foliated by $\{[x, \tilde{m}] |\, \tilde{m} \in \tilde{M}\}$ and G is the restriction to a transversal of the holonomy groupoid of the above foliation.

PROPOSITION 2.5. *If G is a locally compact groupoid, then $G^{(0)}$ is locally closed in G, hence locally compact. If furthermore G is σ-compact, then $G^{(0)}$ is σ-compact.*

Proof. Let Δ be the diagonal in $G \times G$. Since G is locally Hausdorff, Δ is locally closed. Then $G^{(0)} = (\mathrm{Id}, r)^{-1}(\Delta)$ is locally closed in G.
Suppose that $G = \cup_{n \in \mathbb{N}} K_n$ with K_n quasi-compact, then $s(K_n)$ is quasi-compact and $G^{(0)} = \cup_{n \in \mathbb{N}} s(K_n)$. □

PROPOSITION 2.6. *Let Z a locally compact space and G be a locally compact groupoid acting on Z. Then the crossed-product $Z \rtimes G$ is locally compact.*

Proof. Let $p\colon Z \to G^{(0)}$ be the momentum map of the action of G. From Proposition 2.5, the diagonal $\Delta \subset G^{(0)} \times G^{(0)}$ is locally closed in $G^{(0)} \times G^{(0)}$, hence $Z \rtimes G = (p,r)^{-1}(\Delta)$ is locally closed in $Z \times G$. □

Let T be a space. Recall that there is a groupoid $T \times T$ with unit space T, and product $(x,y)(y,z) = (x,z)$.
Let G be a groupoid and T be a space. Let $f\colon T \to G^{(0)}$, and let $G[T] = \{(t',t,g) \in (T\times T)\times G |\ g \in G_{f(t)}^{f(t')}\}$. Then $G[T]$ is a subgroupoid of $(T\times T)\times G$.

PROPOSITION 2.7. *Let G be a topological groupoid with $G^{(0)}$ locally Hausdorff, T a topological space and $f\colon T \to G^{(0)}$ a continuous map. Then $G[T]$ is a locally closed subgroupoid of $(T \times T) \times G$. In particular, if T and G are locally compact, then $G[T]$ is locally compact.*

Proof. Let $F \subset T\times G^{(0)}$ be the graph of f. Then $F = (f\times \mathrm{Id})^{-1}(\Delta)$, where Δ is the diagonal in $G^{(0)} \times G^{(0)}$, thus it is locally closed. Let $\rho\colon (t',t,g) \mapsto (t', r(g))$ and $\sigma\colon (t',t,g) \mapsto (t, s(g))$ be the range and source maps of $(T \times T) \times G$, then $G[T] = (\rho,\sigma)^{-1}(F \times F)$ is locally closed. □

PROPOSITION 2.8. *Let G be a locally compact groupoid such that $G^{(0)}$ is Hausdorff. Then for every $x \in G^{(0)}$, G_x is Hausdorff.*

Proof. Let $Z = \{(g,h) \in G_x \times G_x |\ r(g) = r(h)\}$. Let $\varphi\colon Z \to G$ defined by $\varphi(g,h) = g^{-1}h$. Since $\{x\}$ is closed in G, $\varphi^{-1}(x)$ is closed in Z, and since $G^{(0)}$ is Hausdorff, Z is closed in $G_x \times G_x$. It follows that $\varphi^{-1}(x)$, which is the diagonal of $G_x \times G_x$, is closed in $G_x \times G_x$. □

2.2. PROPER GROUPOIDS.

DEFINITION 2.9. *A topological groupoid G is said to be proper if $(r,s)\colon G \to G^{(0)} \times G^{(0)}$ is proper.*

PROPOSITION 2.10. *Let G be a topological groupoid such that $G^{(0)}$ is locally compact. Consider the following assertions:*

(i) *G is proper;*
(ii) *(r,s) is closed and for every $x \in G^{(0)}$, G_x^x is quasi-compact;*
(iii) *for all quasi-compact subspaces K and L of $G^{(0)}$, G_K^L is quasi-compact;*
(iii)' *for all compact subspaces K and L of $G^{(0)}$, G_K^L is quasi-compact;*
(iv) *for every quasi-compact subspace K of $G^{(0)}$, G_K^K is quasi-compact;*
(v) *$\forall x,\ y \in G^{(0)}$, $\exists K_x$, L_y compact neighborhoods of x and y such that $G_{K_x}^{L_y}$ is quasi-compact.*

Then (i) $\iff$ (ii) $\iff$ (iii) $\iff$ (iii)' $\iff$ (v) $\implies$ (iv). If $G^{(0)}$ is Hausdorff, then (i)–(v) are equivalent.

Proof. (i) $\iff$ (ii) follows from Proposition 1.4, and from the fact that G_x^x is homeomorphic to G_x^y if $G_x^y \neq \emptyset$. (i) $\Longrightarrow$ (iii) and (v) $\Longrightarrow$ (i) follow Proposition 1.6 and the formula $G_K^L = (r,s)^{-1}(L \times K)$. (iii) $\Longrightarrow$ (iii)' $\Longrightarrow$ (v) and (iii) $\Longrightarrow$ (iv) are obvious. If $G^{(0)}$ is Hausdorff, then (iv) $\Longrightarrow$ (v) is obvious. $\square$

Note that if $G = G^{(0)}$ is a non-Hausdorff topological space, then G is not proper (since (r,s) is not closed), but satisfies property (iv).

PROPOSITION 2.11. *Let G be a topological groupoid. If $r\colon G \to G^{(0)}$ is open then the canonical mapping $\pi\colon G^{(0)} \to G^{(0)}/G$ is open.*

Proof. Let $V \subset G^{(0)}$ be an open subspace. If r is open, then $r(s^{-1}(V)) = \pi^{-1}(\pi(V))$ is open. Therefore, $\pi(V)$ is open. $\square$

PROPOSITION 2.12. *Let G be a topological groupoid such that $G^{(0)}$ is locally compact and $r\colon G \to G^{(0)}$ is open. Suppose that $(r,s)(G)$ is locally closed in $G^{(0)} \times G^{(0)}$, then $G^{(0)}/G$ is locally compact. Furthermore,*

(a) *if $G^{(0)}$ is σ-compact, then $G^{(0)}/G$ is σ-compact;*
(b) *if $(r,s)(G)$ is closed (for instance if G is proper), then $G^{(0)}/G$ is Hausdorff.*

Proof. Let $R = (r,s)(G)$. Let $\pi\colon G^{(0)} \to G^{(0)}/G$ be the canonical mapping. By Proposition 2.11, π is open, therefore $G^{(0)}/G$ is locally quasi-compact. Let us show that it is locally Hausdorff. Let V be an open subspace of $G^{(0)}$ such that $(V \times V) \cap R$ is closed in $V \times V$. Let Δ be the diagonal in $\pi(V) \times \pi(V)$. Then $(\pi\times\pi)^{-1}(\Delta) = (V\times V)\cap R$ is closed in $V\times V$. Since $\pi\times\pi\colon V\times V \to \pi(V)\times\pi(V)$ is continuous open surjective, it follows that Δ is closed in $\pi(V) \times \pi(V)$, hence $\pi(V)$ is Hausdorff. This completes the proof that $G^{(0)}/G$ is locally compact and of assertion (b).
Assertion (a) follows from the fact that for every $x \in G^{(0)}$ and every compact neighborhood K of x, $\pi(K)$ is a quasi-compact neighborhood of $\pi(x)$. $\square$

2.3. PROPER ACTIONS.

DEFINITION 2.13. *Let G be a topological groupoid. Let Z be a topological space endowed with an action of G. Then the action is said to be proper if $Z \rtimes G$ is a proper groupoid. (We will also say that Z is a proper G-space.)*

A subspace A of a topological space X is said to be relatively compact (resp. relatively quasi-compact) if it is included in a compact (resp. quasi-compact) subspace of X. This does not imply that $\overline{A}$ is compact (resp. quasi-compact).

PROPOSITION 2.14. *Let G be a topological groupoid. Let Z be a topological space endowed with an action of G. Consider the following assertions:*

(i) *G acts properly on Z;*
(ii) *$(r,s)\colon Z \rtimes G \to Z \times Z$ is closed and $\forall z \in Z$, the stabilizer of z is quasi-compact;*
(iii) *for all quasi-compact subspaces K and L of Z, $\{g \in G|\ Lg \cap K \neq \emptyset\}$ is quasi-compact;*

(iii)' *for all compact subspaces* K *and* L *of* Z, $\{g \in G|\ Lg \cap K \neq \emptyset\}$ *is quasi-compact;*

(iv) *for every quasi-compact subspace* K *of* Z, $\{g \in G|\ Kg \cap K \neq \emptyset\}$ *is quasi-compact;*

(v) *there exists a family* $(A_i)_{i\in I}$ *of subspaces of* Z *such that* $Z = \cup_{i\in I} \mathring{A}_i$ *and* $\{g \in G|\ A_i g \cap A_j \neq \emptyset\}$ *is relatively quasi-compact for all* $i, j \in I$.

Then (i) $\iff$ *(ii)* $\implies$ *(iii)* $\implies$ *(iii)' and (iii)* $\implies$ *(iv). If* Z *is locally compact, then (iii)'* $\implies$ *(v) and (iv)* $\implies$ *(v). If* $G^{(0)}$ *is Hausdorff and* Z *is locally compact Hausdorff, then (i)–(v) are equivalent.*

Proof. (i) $\iff$ (ii) follows from Proposition 2.10[(i) $\iff$ (ii)]. Implication (i) $\implies$ (iii) follows from the fact that if $(Z \rtimes G)_K^L$ is quasi-compact, then its image by the second projection $Z \rtimes G \to G$ is quasi-compact. (iii) $\implies$ (iii)' and (iii) $\implies$ (iv) are obvious.

Suppose that Z is locally compact. Take $A_i \subset Z$ compact such that $Z = \cup_{i\in I} \mathring{A}_i$. If (iii)' is true, then $\{g \in G|\ A_i g \cap A_j \neq \emptyset\}$ is quasi-compact, hence (v). If (iv) is true, then $\{g \in G|\ A_i g \cap A_j \neq \emptyset\}$ is a subset of the quasi-compact set $\{g \in G|\ Kg \cap K \neq \emptyset\}$, where $K = A_i \cup A_j$, hence (v).

Suppose that Z is locally compact Hausdorff and that $G^{(0)}$ is Hausdorff. Let us show (v) $\implies$ (ii). Let C_{ij} be a quasi-compact set such that $\{g \in G|\ A_i g \cap A_j \neq \emptyset\} \subset C_{ij}$.

Let $z \in Z$. Choose $i \in I$ such that $z \in A_i$. Since Z and $G^{(0)}$ are Hausdorff, $\mathrm{stab}(z)$ is a closed subspace of C_{ii}, therefore it is quasi-compact.

It remains to prove that the map $\Phi\colon Z \times_{G^{(0)}} G \to Z \times Z$ given by $\Phi(z, g) = (z, zg)$ is closed. Let $F \subset Z \times_{G^{(0)}} G$ be a closed subspace, and $(z, z') \in \overline{\Phi(F)}$. Choose i and j such that $z \in \mathring{A}_i$ and $z' \in \mathring{A}_j$. Then $(z, z') \in \overline{\Phi(F) \cap (A_i \times A_j)} \subset \overline{\Phi(F \cap (A_i \times_{G^{(0)}} C_{ij}))} \subset \overline{\Phi(F \cap (Z \times_{G^{(0)}} C_{ij}))}$. There exists a net $(z_\lambda, g_\lambda) \in F \cap (Z \times_{G^{(0)}} C_{ij})$ such that (z, z') is a limit point of $(z_\lambda, z_\lambda g_\lambda)$. Since C_{ij} is quasi-compact, after passing to a universal subnet we may assume that g_λ converges to an element $g \in C_{ij}$. Since $G^{(0)}$ is Hausdorff, $F \cap (Z \times_{G^{(0)}} C_{ij})$ is closed in $Z \times C_{ij}$, so (z, g) is an element of $F \cap (Z \times_{G^{(0)}} C_{ij})$. Using the fact that Z is Hausdorff and Φ is continuous, we obtain $(z, z') = \Phi(z, g) \in \Phi(F)$. □

REMARK 2.15. *It is possible to define a notion of slice-proper action which implies properness in the above sense. The two notions are equivalent in many cases* [1, 3].

PROPOSITION 2.16. *Let* G *be a locally compact groupoid. Then* G *acts properly on itself if and only if* $G^{(0)}$ *is Hausdorff. In particular, a locally compact space is proper if and only if it is Hausdorff.*

Proof. It is clear from Proposition 2.10(ii) that G acts properly on itself if and only if the product $\varphi\colon G^{(2)} \to G \times G$ is closed. Since φ factors through the homeomorphism $G^{(2)} \to G \times_{r,r} G$, $(g, h) \mapsto (g, gh)$, G acts properly on itself if and only if $G \times_{r,r} G$ is a closed subset of $G \times G$.

If $G^{(0)}$ is Hausdorff, then clearly $G \times_{r,r} G$ is closed in $G \times G$. Conversely, if $G^{(0)}$ is not Hausdorff, then there exists $(x, y) \in G^{(0)} \times G^{(0)}$ such that $x \neq y$ and (x, y) is in the closure of the diagonal of $G^{(0)} \times G^{(0)}$. It follows that (x, y) is in the closure of $G \times_{r,r} G$, but $(x, y) \notin G \times_{r,r} G$, therefore $G \times_{r,r} G$ is not closed. $\square$

2.4. PERMANENCE PROPERTIES.

PROPOSITION 2.17. *If G_1 and G_2 are proper topological groupoids, then $G_1 \times G_2$ is proper.*

Proof. Follows from the fact that the product of two proper maps is proper [2, Corollaire I.10.2.3]. $\square$

PROPOSITION 2.18. *Let G_1 and G_2 be two topological groupoids such that $G_1^{(0)}$ is Hausdorff and G_2 is proper. Suppose that $f\colon G_1 \to G_2$ is a proper morphism. Then G_1 is proper.*

Proof. Denote by r_i and s_i the range and source maps of G_i $(i = 1, 2)$. Let $\bar{f}$ be the map $G_1^{(0)} \times G_1^{(0)} \to G_2^{(0)} \times G_2^{(0)}$ induced from f. Since $\bar{f} \circ (r_1, s_1) = (r_2, s_2) \circ f$ is proper and $G_1^{(0)}$ is Hausdorff, it follows from [2, Proposition I.10.1.5] that (r_1, s_1) is proper. $\square$

PROPOSITION 2.19. *Let G_1 and G_2 be two topological groupoids such that G_1 is proper. Suppose that $f\colon G_1 \to G_2$ is a surjective morphism such that the induced map $f'\colon G_1^{(0)} \to G_2^{(0)}$ is proper. Then G_2 is proper.*

Proof. Denote by r_i and s_i the range and source maps of G_i $(i = 1, 2)$. Let $F_2 \subset G_2$ be a closed subspace, and $F_1 = f^{-1}(F_2)$. Since G_1 is proper, $(r_1, s_1)(F_1)$ is closed, and since $f' \times f'$ is proper, $(f' \times f') \circ (r_1, s_1)(F_1)$ is closed. By surjectivity of f, we have $(r_2, s_2)(F_2) = (f' \times f') \circ (r_1, s_1)(F_1)$. This proves that (r_2, s_2) is closed. Since for every topological space T, the assumptions of the proposition are also true for the morphism $f \times 1\colon G_1 \times T \to G_2 \times T$, the above shows that $(r_2, s_2) \times 1_T$ is closed. Therefore, (r_2, s_2) is proper. $\square$

PROPOSITION 2.20. *Let G be a topological groupoid with $G^{(0)}$ Hausdorff, acting on two spaces Y and Z. Suppose that the action of G on Z is proper, and that Y is Hausdorff. Then G acts properly on $Y \times_{G^{(0)}} Z$.*

Proof. The groupoid $(Y \times_{G^{(0)}} Z) \rtimes G$ is isomorphic to the subgroupoid $\Gamma = \{(y, y', z, g) \in (Y \times Y) \times (Z \rtimes G) |\, p(y) = r(g),\ y' = yg\}$ of the proper groupoid $(Y \times Y) \times (Z \rtimes G)$. Since Y and $G^{(0)}$ are Hausdorff, Γ is closed in $(Y \times Y) \times (Z \rtimes G)$, hence by Proposition 2.10(ii), $(Y \times_{G^{(0)}} Z) \rtimes G$ is proper. $\square$

COROLLARY 2.21. *Let G be a proper topological groupoid with $G^{(0)}$ Hausdorff. Then any action of G on a Hausdorff space is proper.*

Proof. Follows from Proposition 2.20 with $Z = G^{(0)}$. $\square$

PROPOSITION 2.22. *Let G be a topological groupoid and $f\colon T \to G^{(0)}$ be a continuous map.*

(a) *If G is proper, then $G[T]$ is proper.*
(ii) *If $G[T]$ is proper and f is open surjective, then G is proper.*

Proof. Let us prove (a). Suppose first that T is a subspace of $G^{(0)}$ and that f is the inclusion. Then $G[T] = G_T^T$. Since (r_T, s_T) is the restriction to $(r,s)^{-1}(T\times T)$ of (r,s), and (r,s) is proper, it follows that (r_T, s_T) is proper. In the general case, let $\Gamma = (T\times T)\times G$ and let $T' \subset T\times G^{(0)}$ be the graph of f. Then Γ is a proper groupoid (since it is the product of two proper groupoids), and $G[T] = \Gamma[T']$.

Let us prove (b). The only difficulty is to show that (r,s) is closed. Let $F\subset G$ be a closed subspace and $(y,x)\in \overline{(r,s)(F)}$. Let $\tilde F = G[T]\cap (T\times T)\times F$. Choose $(t',t)\in T\times T$ such that $f(t') = y$ and $f(t) = x$. Denote by $\tilde r$ and $\tilde s$ the range and source maps of $G[T]$. Then $(t',t)\in \overline{(\tilde r,\tilde s)(\tilde F)}$. Indeed, let $\Omega \ni (t',t)$ be an open set, and $\Omega' = (f\times f)(\Omega)$. Then Ω' is an open neighborhood of (y,x), so $\Omega'\cap (r,s)(F)\neq\emptyset$. It follows that $\Omega\cap(\tilde r,\tilde s)(\tilde F)\neq\emptyset$.
We have proved that $(t',t)\in\overline{(\tilde r,\tilde s)(\tilde F)} = (\tilde r,\tilde s)(\tilde F)$, so $(y,x)\in (r,s)(F)$. □

COROLLARY 2.23. *Let G be a groupoid acting properly on a topological space Z, and let Z_1 be a saturated subspace. Then G acts properly on Z_1.*

Proof. Use the fact that $Z_1 \rtimes G = (Z\rtimes G)[Z_1]$. □

2.5. INVARIANCE BY MORITA-EQUIVALENCE. In this section, we will only consider groupoids whose range maps are open. We thus need a stability lemma:

LEMMA 2.24. *Let G be a topological groupoid whose range map is open. Let Z be a G space and $f\colon T\to G^{(0)}$ be a continuous open map. Then the range maps for $Z\rtimes G$ and $G[T]$ are open.*

To prove Lemma 2.24 we need a preliminary result:

LEMMA 2.25. *Let X, Y, T be topological spaces, $g\colon Y\to T$ an open map and $f\colon X\to T$ continuous. Let $Z = X\times_T Y$. Then the first projection $pr_1\colon X\times_T Y\to X$ is open.*

Proof. Let $\Omega\subset Z$ open. There exists an open subspace Ω' of $X\times Y$ such that $\Omega = \Omega'\cap Z$. Let Δ be the diagonal in $X\times X$. One easily checks that $(\mathrm{pr}_1,\mathrm{pr}_1)(\Omega) = (1\times f)^{-1}(1\times g)(\Omega')\cap\Delta$, therefore $(\mathrm{pr}_1,\mathrm{pr}_1)(\Omega)$ is open in Δ. This implies that $\mathrm{pr}_1(\Omega)$ is open in X. □

Proof of Lemma 2.24. This is clear for $Z\rtimes G = Z\times_{G^{(0)}} G$ using Lemma 2.25. For $G[T]$, first use Lemma 2.25 to prove that $T\times_{f,s} G\xrightarrow{pr_2} G$ is open. Since the range map is open by assumption, the composition $T\times_{f,s}G\xrightarrow{pr_2} G\xrightarrow{r} G^{(0)}$ is open. Using again Lemma 2.25, $G[T]\simeq T\times_{f,r\circ pr_2}(T\times_{f,s}G)\xrightarrow{pr_1} T$ is open. □

In order to define the notion of Morita-equivalence for topological groupoids, we introduce some terminology:

DEFINITION 2.26. *Let G be a topological groupoid. Let T be a topological space and $\rho\colon G^{(0)} \to T$ be a G-invariant map. Then G is said to be ρ-proper if the map $(r,s)\colon G \to G^{(0)} \times_T G^{(0)}$ is proper. If G acts on a space Z and $\rho\colon Z \to T$ is G-invariant, then the action is said to be ρ-proper if $Z \rtimes G$ is ρ-proper.*

It is clear that properness implies ρ-properness. There is a partial converse:

PROPOSITION 2.27. *Let G be a topological groupoid, T a topological space, $\rho\colon G^{(0)} \to T$ a G-invariant map. If G is ρ-proper and T is Hausdorff, then G is proper.*

Proof. Since T is Hausdorff, $G^{(0)} \times_T G^{(0)}$ is a closed subspace of $G^{(0)} \times G^{(0)}$, therefore (r,s), being the composition of the two proper maps $G \to G^{(0)} \times_T G^{(0)} \to G^{(0)} \times G^{(0)}$, is proper. □

REMARK 2.28. *When T is locally Hausdorff, one easily shows that G is ρ-proper iff for every Hausdorff open subspace V of T, $G_{\rho^{-1}(V)}^{\rho^{-1}(V)}$ is proper.*

PROPOSITION 2.29. [14] *Let G_1 and G_2 be two topological (resp. locally compact) groupoids. Let r_i, s_i $(i = 1,2)$ be the range and source maps of G_i, and suppose that r_i are open. The following are equivalent:*

(i) *there exist a topological (resp. locally compact) space T and $f_i\colon T \to G_i^{(0)}$ open surjective such that $G_1[T]$ and $G_2[T]$ are isomorphic;*

(ii) *there exists a topological (resp. locally compact) space Z, two continuous maps $\rho\colon Z \to G_1^{(0)}$ and $\sigma\colon Z \to G_2^{(0)}$, a left action of G_1 on Z with momentum map ρ and a right action of G_2 on Z with momentum map σ such that*
   (a) *the actions commute and are free, the action of G_2 is ρ-proper and the action of G_1 is σ-proper;*
   (b) *the natural maps $Z/G_2 \to G_1^{(0)}$ and $G_1\backslash Z \to G_2^{(0)}$ induced from ρ and σ are homeomorphisms.*

Moreover, one may replace (b) by

(b)' *ρ and σ are open and induce bijections $Z/G_2 \to G_1^{(0)}$ and $G_1\backslash Z \to G_2^{(0)}$.*
In (i)*, if T is locally compact then it may be assumed Hausdorff.*

If G_1 and G_2 satisfy the equivalent conditions in Proposition 2.29, then they are said to be Morita-equivalent. Note that if $G_i^{(0)}$ are Hausdorff, then by Proposition 2.27, one may replace "ρ-proper" and "σ-proper" by "proper".
To prove Proposition 2.29, we need preliminary lemmas:

LEMMA 2.30. *Let G be a topological groupoid. The following are equivalent:*

(i) *$r\colon G \to G^{(0)}$ is open;*
(ii) *for every G-space Z, the canonical mapping $\pi\colon Z \to Z/G$ is open.*

Proof. To show (ii) $\Longrightarrow$ (i), take $Z = G$: the canonical mapping $\pi\colon G \to G/G$ is open. Therefore, for every open subspace U of G, $r(U) = G^{(0)} \cap \pi^{-1}(\pi(U))$ is open.
Let us show (i) $\Longrightarrow$ (ii). By Lemma 2.24, the range map $r\colon Z \rtimes G \to Z$ is open. The conclusion follows from Proposition 2.11. $\square$

LEMMA 2.31. *Let G be a topological groupoid such that the range map $r\colon G \to G^{(0)}$ is open. Let X be a topological space endowed with an action of G and T a topological space. Then the canonical map*

$$f\colon (X \times T)/G \to (X/G) \times T$$

is an isomorphism.

Proof. Let $\pi\colon X \to X/G$ and $\pi'\colon X \times T \to (X \times T)/G$ be the canonical mappings. Since π is open (Lemma 2.30), $f \circ \pi' = \pi \times 1$ is open. Since π' is continuous surjective, it follows that f is open. $\square$

LEMMA 2.32. *Let G be a topological groupoid whose range map is open and $f\colon Y \to Z$ a proper, G-equivariant map between two G-spaces. Then the induced map $\bar{f}\colon Y/G \to Z/G$ is proper.*

Proof. We first show that $\bar{f}$ is closed. Let $\pi\colon Y \to Y/G$ and $\pi'\colon Z \to Z/G$ be the canonical mappings. Let $A \subset Y/G$ be a closed subspace. Since f is closed and π is continuous, $(\pi')^{-1}(\bar{f}(A)) = f(\pi^{-1}(A))$ is closed. Therefore, $\bar{f}(A)$ is closed.
Applying this to $f \times 1$, we see that for every topological space T, $(Y \times T)/G \to (Z \times T)/G$ is closed. By Lemma 2.31, $\bar{f} \times 1_T$ is closed. $\square$

LEMMA 2.33. *Let G_2 and G_3 be topological groupoids whose range maps are open. Let Z_1, Z_2 and X be topological spaces. Suppose there are maps*

$$X \xleftarrow{\rho_1} Z_1 \xrightarrow{\sigma_1} G_2^{(0)} \xleftarrow{\rho_2} Z_2 \xrightarrow{\sigma_2} G_3^{(0)},$$

a right action of G_2 on Z_1 with momentum map σ_1, such that ρ_1 is G_2-invariant and the action of G_2 is ρ_1-proper, a left action of G_2 on Z_2 with momentum map ρ_2 and a right ρ_2-proper action of G_3 on Z_2 with momentum map σ_2 which commutes with the G_2-action.
Then the action of G_3 on $Z = Z_1 \times_{G_2} Z_2$ is ρ_1-proper.

Proof. Let $\varphi\colon Z_2 \rtimes G_3 \to Z_2 \times_{G_2^{(0)}} Z_2$ be the map $(z_2, \gamma) \mapsto (z_2, z_2\gamma)$. By assumption, φ is proper, therefore $1_{Z_1} \times \varphi$ is proper. Let $F = \{(z_1, z_2, z_2') \in Z_1 \times Z_2 \times Z_2 |\ \sigma_1(z_1) = \rho_2(z_2) = \rho_2(z_2')\}$. Then $1_{Z_1} \times \varphi\colon (1 \times \varphi)^{-1}(F) \to F$ is proper, i.e. $Z_1 \times_{G_2^{(0)}} (Z_2 \rtimes G_3) \to Z_1 \times_{G_2^{(0)}} (Z_2 \times_{G_2^{(0)}} Z_2)$ is proper. By Lemma 2.32, taking the quotient by G_2, we get that the map

$$\alpha\colon Z \rtimes G_3 \to Z_1 \times_{G_2} (Z_2 \times_{G_2^{(0)}} Z_2)$$

defined by $(z_1, z_2, \gamma) \mapsto (z_1, z_2, z_2\gamma)$ is proper.

By assumption, the map $Z_1 \rtimes G_2 \to Z_1 \times_X Z_1$ given by $(z_1, g) \mapsto (z_1, z_1 g)$ is proper. Endow $Z_1 \rtimes G_2$ with the following right action of $G_2 \times G_2$: $(z_1, g) \cdot (g', g'') = (z_1 g', (g')^{-1} g g'')$. Using again Lemma 2.32, the map

$$\begin{aligned}\beta\colon Z_1 \times_{G_2} (Z_2 \times_{G_2^{(0)}} Z_2) &= (Z_1 \rtimes G_2) \times_{G_2 \times G_2} (Z_2 \times Z_2)\\ &\to (Z_1 \times_X Z_1) \times_{G_2 \times G_2} (Z_2 \times Z_2) \simeq Z \times_X Z\end{aligned}$$

is proper. By composition, $\beta \circ \alpha\colon Z \rtimes G_3 \to Z \times_X Z$ is proper. □

Proof of Proposition 2.29. Let us treat the case of topological groupoids. Assertion (b') follows from the fact that the canonical mappings $Z \to Z/G_2$ and $Z \to G_1 \backslash Z$ are open (Lemma 2.30).

Let us first show that (ii) is an equivalence relation. Reflexivity is clear (taking $Z = G$, $\rho = r$, $\sigma = s$), and symmetry is obvious. Suppose that (Z_1, ρ_1, σ_2) and (Z_2, ρ_2, σ_2) are equivalences between G_1 and G_2, and G_2 and G_3 respectively. Let $Z = Z_1 \times_{G_2} Z_2$ be the quotient of $Z_1 \times_{G_2^{(0)}} Z_2$ by the action $(z_1, z_2) \cdot \gamma = (z_1 \gamma, \gamma^{-1} z_2)$ of G_2. Denote by $\rho\colon Z \to G_1^{(0)}$ and $\sigma\colon Z \to G_3^{(0)}$ the maps induced from $\rho_1 \times 1$ and $1 \times \sigma_2$. By Lemma 2.25, the first projection $pr_1\colon Z_1 \times_{G_2^{(0)}} Z_2 \to Z_1$ is open, therefore $\rho = \rho_1 \circ pr_1$ is open. Similarly, σ is open. It remains to show that the actions of G_3 and G_1 are ρ-proper and σ-proper respectively. For G_3, this follows from Lemma 2.33 and the proof for G_1 is similar.

This proves that (ii) is an equivalence relation. Now, let us prove that (i) and (ii) are equivalent.

Suppose (ii). Let $\Gamma = G_1 \ltimes Z \rtimes G_2$ and $T = Z$. The maps $\rho\colon T \to G_1^{(0)}$ and $\sigma\colon T \to G_2^{(0)}$ are open surjective by assumption. Since $G_1 \ltimes Z \simeq Z \times_{G_2^{(0)}} Z$ and $Z \rtimes G_2 \simeq Z \times_{G_1^{(0)}} Z$, we have $G_2[T] = (T \times T) \times_{G_2^{(0)} \times G_2^{(0)}} G_2 \simeq (Z \rtimes G_2) \times_{s \circ pr_2, \sigma} Z \simeq (Z \times_{G_1^{(0)}} Z) \times_{\sigma \circ pr_2, \sigma} Z = Z \times_{G_1^{(0)}} (Z \times_{G_2^{(0)}} Z) \simeq Z \times_{G_1^{(0)}} (G_1 \ltimes Z) \simeq G_1 \ltimes (Z \times_{G_1^{(0)}} Z) \simeq G_1 \ltimes (Z \rtimes G_2) = \Gamma$. Similarly, $\Gamma \simeq G_1[T]$, hence (i).

Conversely, to prove $(i) \implies (ii)$ it suffices to show that if $f\colon T \to G^{(0)}$ is open surjective, then G and $G[T]$ are equivalent in the sense (ii), since we know that (ii) is an equivalence relation. Let $Z = T \times_{r,f} G$.

Let us check that the action of G is pr_1-proper. Write $Z \rtimes G = \{(t, g, h) \in T \times G \times G |\ f(t) = r(g) \text{ and } s(g) = r(h)\}$. One needs to check that the map $Z \rtimes G \to (T \times_{f,r} G)^2$ defined by $(t, g, h) \mapsto (t, g, t, h)$ is a homeomorphism onto its image. This follows easily from the facts that the diagonal map $T \to T \times T$ and the map $G^{(2)} \to G \times G$, $(g, h) \mapsto (g, gh)$ are homeomorphisms onto their images.

Let us check that the action of $G[T]$ is $s \circ pr_2$-proper. One easily checks that the groupoid $G' = G[T] \ltimes (T \times_{f,r} G)$ is isomorphic to a subgroupoid of the trivial groupoid $(T \times T) \times (G \times G)$. It follows that if r' and s' denote the range and source maps of G', the map (r', s') is a homeomorphism of G' onto its image.

Let us now treat the case of locally compact groupoids. In the proof that (ii) is a transitive relation, it just remains to show that Z is locally compact.

Let U_3 be a Hausdorff open subspace of $G_3^{(0)}$. We show that $\sigma^{-1}(U_3)$ is locally compact. Replacing G_3 by $(G_3)_{U_3}^{U_3}$, we may assume that G_2 acts freely and properly on Z_2. Let Γ be the groupoid $(Z_1 \times_{G_2^{(0)}} Z_2) \rtimes G_2$, and $R = (r,s)(\Gamma) \subset (Z_1 \times_{G_2^{(0)}} Z_2)^2$. Since the action of G_2 on Z_2 is free and proper, there exists a continuous map $\varphi\colon Z_2 \times_{G_3^{(0)}} Z_2 \to G_2$ such that $z_2 = \varphi(z_2, z_2')z_2'$. Then $R = \{(z_1, z_2, z_1', z_2') \in (Z_1 \times_{G_2^{(0)}} Z_2)^2;\ z_1' = z_1\varphi(z_2, z_2')\}$ is locally closed. By Proposition 2.12, $Z = (Z_1 \times_{G_2^{(0)}} Z_2)/G$ is locally compact.

Finally, if (i) holds with $T = \cup_i V_i$ with V_i open Hausdorff, let $T' = \amalg V_i$. It is clear that $G_1[T'] \simeq G_2[T']$. ☐

Let us examine standard examples of Morita-equivalences:

EXAMPLE 2.34. *Let G be a topological groupoid whose range map is open. Let $(U_i)_{i\in I}$ be an open cover of $G^{(0)}$ and $\mathcal{U} = \amalg_{i\in I} U_i$. Then $G[\mathcal{U}]$ is Morita-equivalent to G.*

EXAMPLE 2.35. *Let G be a topological groupoid, and let H_1, H_2 be subgroupoids such that the range maps $r_i\colon H_i \to H_i^{(0)}$ are open. Then $(H_1\backslash G_{s(H_2)}^{s(H_1)}) \rtimes H_2$ and $H_1 \ltimes (G_{s(H_2)}^{s(H_1)}/H_2)$ are Morita-equivalent.*

Proof. Take $Z = G_{s(H_2)}^{s(H_1)}$ and let $\rho\colon Z \to Z/H_2$ and $\sigma\colon H_1\backslash Z$ be the canonical mappings. The fact that these maps are open follows from Lemma 2.30. ☐

The following proposition is an immediate consequence of Proposition 2.22.

PROPOSITION 2.36. *Let G and G' be two topological groupoids such that the range maps of G and G' are open. Suppose that G and G' are Morita-equivalent. Then G is proper if and only if G' is proper.*

COROLLARY 2.37. *With the notations of Example 2.34, G is proper if and only if $G[\mathcal{U}]$ is proper.*

3. A TOPOLOGICAL CONSTRUCTION

Let X be a locally compact space. Since X is not necessarily Hausdorff, a filter[1] $\mathcal{F}$ on X may have more than one limit. Let S be the set of limits of a convergent filter $\mathcal{F}$. The goal of this section is to construct a Hausdorff space $\mathcal{H}X$ in which X is (not continuously) embedded, and such that $\mathcal{F}$ converges to S in $\mathcal{H}X$.

3.1. THE SPACE $\mathcal{H}X$.

LEMMA 3.1. *Let X be a topological space, and $S \subset X$. The following are equivalent:*

(i) *for every family $(V_s)_{s\in S}$ of open sets such that $s \in V_s$, and $V_s = X$ except perhaps for finitely many s's, one has $\cap_{s\in S} V_s \neq \emptyset$;*

[1] or a net; we will use indifferently the two equivalent approaches

(ii) *for every finite family* $(V_i)_{i\in I}$ *of open sets such that* $S\cap V_i \neq \emptyset$ *for all* i*, one has* $\cap_{i\in I} V_i \neq \emptyset$.

Proof. (i) $\Longrightarrow$ (ii): let $(V_i)_{i\in I}$ as in (ii). For all i, choose $s(i)\in S\cap V_i$. Put $W_s = \cap_{s=s(i)} V_i$, with the convention that an empty intersection is X. Then by (i), $\emptyset \neq \cap_{s\in S} W_s = \cap_{i\in I} V_i$.
(ii) $\Longrightarrow$ (i): let $(V_s)_{s\in S}$ as in (i), and let $I = \{s\in S|\ V_s \neq X\}$. Then $\cap_{s\in S} V_s = \cap_{i\in I} V_i \neq \emptyset$. □

We shall denote by $\mathcal{H}X$ the set of non-empty subspaces S of X which satisfy the equivalent conditions of Lemma 3.1, and $\hat{\mathcal{H}}X = \mathcal{H}X \cup \{\emptyset\}$.

LEMMA 3.2. *Let* X *be a locally Hausdorff space. Then every* $S\in\mathcal{H}X$ *is locally finite. More precisely, if* V *is a Hausdorff open subspace of* X*, then* $V\cap S$ *has at most one element.*

Proof. Suppose $a\neq b$ and $\{a,b\}\subset V\cap S$. Then there exist V_a, V_b open disjoint neighborhoods of a and b respectively; this contradicts Lemma 3.1(ii). □

Suppose that X is locally compact. We endow $\hat{\mathcal{H}}X$ with a topology. Let us introduce the notations $\Omega_V = \{S\in\mathcal{H}X|\ V\cap S\neq\emptyset\}$ and $\Omega^Q = \{S\in \mathcal{H}X|\ Q\cap S = \emptyset\}$. The topology on $\hat{\mathcal{H}}X$ is generated by the Ω_V's and Ω^Q's (V open and Q quasi-compact). More explicitly, a set is open if and only if it is a union of sets of the form $\Omega^Q_{(V_i)_{i\in I}} = \Omega^Q \cap (\cap_{i\in I}\Omega_{V_i})$ where $(V_i)_{i\in I}$ is a finite family of open Hausdorff sets and Q is quasi-compact.

PROPOSITION 3.3. *For every locally compact space* X*, the space* $\hat{\mathcal{H}}X$ *is Hausdorff.*

Proof. Suppose $S\not\subset S'$ and S, $S'\in\hat{\mathcal{H}}X$. Let $s\in S-S'$. Since S' is locally finite and since every singleton subspace of X is closed, there exist V open and K compact such that $s\in V\subset K$ and $K\cap S'=\emptyset$. Then Ω_V and Ω^K are disjoint neighborhoods of S and S' respectively. □

For every filter $\mathcal{F}$ on $\hat{\mathcal{H}}X$, let

$$L(\mathcal{F}) = \{a\in X|\ \forall V\ni a \text{ open}, \Omega_V\in\mathcal{F}\}. \tag{1}$$

LEMMA 3.4. *Let* X *be a locally compact space. Let* $\mathcal{F}$ *be a filter on* $\hat{\mathcal{H}}X$*. Then* $\mathcal{F}$ *converges to* $S\in\hat{\mathcal{H}}X$ *if and only if properties (a) and (b) below hold:*

(a) $\forall V$ *open,* $V\cap S\neq\emptyset \implies \Omega_V\in\mathcal{F}$*;*
(b) $\forall Q$ *quasi-compact,* $Q\cap S=\emptyset \implies \Omega^Q\in\mathcal{F}$.

If $\mathcal{F}$ *is convergent, then* $L(\mathcal{F})$ *is its limit.*

Proof. The first statement is obvious, since every open set in $\hat{\mathcal{H}}X$ is a union of finite intersections of Ω_V's and Ω^Q's.
Let us prove the second statement. It is clear from (a) that $S\subset L(\mathcal{F})$. Conversely, suppose there exists $a\in L(\mathcal{F})-S$. Since S is locally finite and every singleton subspace of X is closed, there exists a compact neighborhood K of a such that $K\cap S=\emptyset$. Then $a\in L(\mathcal{F})$ implies $\Omega_K\in\mathcal{F}$, and condition (b)

implies $\Omega^K \in \mathcal{F}$, thus $\emptyset = \Omega^K \cap \Omega_K \in \mathcal{F}$, which is impossible: we have proved the reverse inclusion $L(\mathcal{F}) \subset S$. $\square$

REMARK 3.5. *This means that if $S_\lambda \to S$, then $a \in S$ if and only if $\forall \lambda$ there exists $s_\lambda \in S_\lambda$ such that $s_\lambda \to a$.*

EXAMPLE 3.6. *Consider Example 2.3 with $\Gamma = \mathbb{Z}_2$ and $H = \{0\}$. Then $\mathcal{H}G = G \cup \{S\}$ where $S = \{(0,0),(0,1)\}$. The sequence $(1/n, 0) \in G$ converges to S in $\mathcal{H}G$, and $(0,0)$ and $(0,1)$ are two isolated points in $\mathcal{H}G$.*

PROPOSITION 3.7. *Let X be a locally compact space and $K \subset X$ quasi-compact. Then $L = \{S \in \mathcal{H}X |\ S \cap K \neq \emptyset\}$ is compact. The space $\mathcal{H}X$ is locally compact, and it is σ-compact if X is σ-compact.*

Proof. We show that L is compact, and the two remaining assertions follow easily. Let $\mathcal{F}$ be a ultrafilter on L. Let $S_0 = L(\mathcal{F})$. Let us show that $S_0 \cap K \neq \emptyset$: for every $S \in L$, choose a point $\varphi(S) \in K \cap S$. By quasi-compactness, $\varphi(\mathcal{F})$ converges to a point $a \in K$, and it is not hard to see that $a \in S_0$.
Let us show $S_0 \in \mathcal{H}X$: let (V_s) $(s \in S_0)$ be a family of open subspaces of X such that $s \in V_s$ for all $s \in S_0$, and $V_s = X$ for every $s \notin S_1$ ($S_1 \subset S_0$ finite). By definition of S_0, $\Omega_{(V_s)_{s\in S_1}} = \cap_{s \in S_1} \Omega_{V_s}$ belongs to $\mathcal{F}$, hence it is non-empty. Choose $S \in \Omega_{(V_s)_{s\in S_1}}$, then $S \cap V_s \neq \emptyset$ for all $s \in S_1$. By Lemma 3.1(ii), $\cap_{s\in S_1} V_s \neq \emptyset$. This shows that $S_0 \in \mathcal{H}X$.
Now, let us show that $\mathcal{F}$ converges to S_0.

- If V is open Hausdorff such that $S_0 \in \Omega_V$, then by definition $\Omega_V \in \mathcal{F}$.
- If Q is quasi-compact and $S_0 \in \Omega^Q$, then $\Omega^Q \in \mathcal{F}$, otherwise one would have $\{S \in \mathcal{H}X |\ S \cap Q \neq \emptyset\} \in \mathcal{F}$, which would imply as above that $S_0 \cap Q \neq \emptyset$, a contradiction.

From Lemma 3.4, $\mathcal{F}$ converges to S_0. $\square$

PROPOSITION 3.8. *Let X be a locally compact space. Then $\hat{\mathcal{H}}X$ is the one-point compactification of $\mathcal{H}X$.*

Proof. It suffices to prove that $\hat{\mathcal{H}}X$ is compact. The proof is almost the same as in Proposition 3.7. $\square$

REMARK 3.9. *If $f \colon X \to Y$ is a continuous map from a locally compact space X to any* Hausdorff *space Y, then f induces a continuous map $\mathcal{H}f \colon \mathcal{H}X \to Y$. Indeed, for every open subspace V of Y, $(\mathcal{H}f)^{-1}(V) = \Omega_{f^{-1}(V)}$ is open.*

PROPOSITION 3.10. *Let G be a topological groupoid such that $G^{(0)}$ is Hausdorff, and $r \colon G \to G^{(0)}$ is open. Let Z be a locally compact space endowed with a continuous action of G. Then $\mathcal{H}Z$ is endowed with a continuous action of G which extends the one on Z.*

Proof. Let $p \colon Z \to G^{(0)}$ such that G acts on Z with momentum map p. Since p has a continuous extension $\mathcal{H}p \colon \mathcal{H}Z \to G^{(0)}$, for all $S \in \mathcal{H}Z$, there exists $x \in G^{(0)}$ such that $S \subset p^{-1}(x)$. For all $g \in G^x$, write $Sg = \{sg |\ s \in S\}$.

Let us show that $Sg \in \mathcal{H}Z$. Let V_s $(s \in S)$ be open sets such that $sg \in V_s$. By continuity, there exist open sets $W_s \ni s$ and $W_g \ni g$ such that for all $(z, h) \in W_s \times_{G^{(0)}} W_g$, $zh \in V_s$. Let $V'_s = W_s \cap p^{-1}(r(W_g))$. Then V'_s is an open neighborhood of s, so there exists $z \in \cap_{s\in S} V'_s$. Since $p(z) \in r(W_g)$, there exists $h \in W_g$ such that $p(z) = r(h)$. It follows that $zh \in \cap_{s\in S} V_s$. This shows that $Sg \in \mathcal{H}Z$.

Let us show that the action defined above is continuous. Let $\Phi\colon \mathcal{H}Z \times_{G^{(0)}} G \to \mathcal{H}Z$ be the action of G on $\mathcal{H}Z$. Suppose that $(S_\lambda, g_\lambda) \to (S, g)$ and let $S' = L((S_\lambda, g_\lambda))$. Then for all $a \in S$ there exists $s_\lambda \in S_\lambda$ such that $s_\lambda \to a$. This implies $s_\lambda g_\lambda \to ag$, thus $ag \in S'$. The converse may be proved in a similar fashion, hence $Sg = S'$.

Applying this to any universal net (S_λ, g_λ) converging to (S, g) and knowing from Proposition 3.8 that $\Phi(S_\lambda, g_\lambda)$ is convergent in $\hat{\mathcal{H}}Z$, we find that $\Phi(S_\lambda, g_\lambda)$ converges to $\Phi(S, g)$. This shows that Φ is continuous in (S, g).

□

3.2. THE SPACE $\mathcal{H}'X$. Let X be a locally compact space. Let $\Omega'_V = \{S \in \mathcal{H}X |\ S \subset V\}$. Let $\mathcal{H}'X$ be $\mathcal{H}X$ as a set, with the coarsest topology such that the identity map $\mathcal{H}'X \to \mathcal{H}X$ is continuous, and Ω'_V is open for every relatively quasi-compact open set V. The space $\mathcal{H}'X$ is Hausdorff since $\mathcal{H}X$ is Hausdorff, but it is usually not locally compact.

LEMMA 3.11. *Let X be a locally compact space. Then the map*

$$\mathcal{H}'X \to \mathbb{N}^* \cup \{\infty\}, \quad S \mapsto \#S$$

is upper semi-continuous.

Proof. Let $S \in \mathcal{H}'X$ such that $\#S < \infty$. Let V_s $(s \in S)$ be open relatively compact Hausdorff sets such that $s \in V_s$, and let $W = \cup_{s\in S} V_s$. Then $S' \in \mathcal{H}'X$ implies $\#(S' \cap V_s) \le 1$, therefore $S' \in \Omega'_W$ implies $\#S' \le \#S$. □

PROPOSITION 3.12. *Let X be a locally compact space such that the closure of every quasi-compact subspace is quasi-compact. Then*

(a) *the natural map $\mathcal{H}'X \to \mathcal{H}X$ is a homeomorphism,*
(b) *for every compact subspace $K \subset X$, there exists $C_K > 0$ such that*

$$\forall S \in \mathcal{H}X,\ S \cap K \neq \emptyset \implies \#S \le C_K,$$

(c) *If G is a locally compact proper groupoid with $G^{(0)}$ Hausdorff then G satisfies the above properties.*

Proof. To prove (b), let K_1 be a quasi-compact neighborhood of K and let $K' = \overline{K}_1$. Let $a \in K \cap S$ and suppose there exists $b \in S - K'$. Then $\mathring{K}_1$ and $X - K'$ are disjoint neighborhoods of a and b respectively, which is impossible. We deduce that $S \subset K'$.

Now, let $(V_i)_{i\in I}$ be a finite cover of K' by open Hausdorff sets. For all $b \in S$, let $I_b = \{i \in I |\ b \in V_i\}$. By Lemma 3.2, the I_b's $(b \in S)$ are disjoint, whence one may take $C_K = \#I$.

To prove (a), denote by $\Delta \subset X \times X$ the diagonal. Let us first show that $pr_1 \colon \overline{\Delta} \to X \times X$ is proper.
Let $K \subset X$ compact. Let $L \subset X$ quasi-compact such that $K \subset \mathring{L}$. If $(a,b) \in \overline{\Delta} \cap (K \times X)$, then $b \in \overline{L}$: otherwise, $L \times L^c$ would be a neighborhood of (a,b) whose intersection with Δ is empty. Therefore, $pr_1^{-1}(K) = \overline{\Delta} \cap (K \times \overline{L})$ is quasi-compact, which shows that pr_1 is proper.
It remains to prove that Ω'_V is open in $\mathcal{H}X$ for every relatively quasi-compact open set $V \subset X$. Let $S \in \Omega'_V$, $a \in S$ and K a compact neighborhood of a. Let $L = pr_2(\overline{\Delta} \cap (K \times X))$. Then $Q = L - V$ is quasi-compact, and $S \in \Omega^Q_{\mathring{K}} \subset \Omega'_V$, therefore Ω'_V is a neighborhood of each of its points.

To prove (c), let $K \subset G$ be a quasi-compact subspace. Then $L = r(K) \cup s(K)$ is quasi-compact, thus G^L_L is also quasi-compact. But $\overline{K}$ is closed and $\overline{K} \subset G^L_L$, therefore $\overline{K}$ is quasi-compact. □

4. Haar systems

4.1. The space $C_c(X)$. For every locally compact space X, $C_c(X)_0$ will denote the set of functions $f \in C_c(V)$ (V open Hausdorff), extended by 0 outside V. Let $C_c(X)$ be the linear span of $C_c(X)_0$. Note that functions in $C_c(X)$ are not necessarily continuous.

Proposition 4.1. *Let X be a locally compact space, and let $f \colon X \to \mathbb{C}$. The following are equivalent:*

(i) *$f \in C_c(X)$;*
(ii) *$f^{-1}(\mathbb{C}^*)$ is relatively quasi-compact, and for every filter $\mathcal{F}$ on X, let $\tilde{\mathcal{F}} = i(\mathcal{F})$, where $i \colon X \to \mathcal{H}X$ is the canonical inclusion; if $\tilde{\mathcal{F}}$ converges to $S \in \mathcal{H}X$, then $\lim_{\mathcal{F}} f = \sum_{s \in S} f(s)$.*

Proof. Let us show (i) $\Longrightarrow$ (ii). By linearity, it is enough to consider the case $f \in C_c(V)$, where $V \subset X$ is open Hausdorff. Let K be the compact set $\overline{f^{-1}(\mathbb{C}^*)} \cap V$. Then $f^{-1}(\mathbb{C}^*) \subset K$. Let $\mathcal{F}$ and S as in (ii). If $S \cap V = \emptyset$, then $S \in \Omega^K$, hence $\Omega^K \in \tilde{\mathcal{F}}$, i.e. $X - K \in \mathcal{F}$. Therefore, $\lim_{\mathcal{F}} f = 0 = \sum_{s \in S} f(s)$. If $S \cap V = \{a\}$, then a is a limit point of $\mathcal{F}$, therefore $\lim_{\mathcal{F}} f = f(a) = \sum_{s \in S} f(s)$.
Let us show (ii) $\Longrightarrow$ (i) by induction on $n \in \mathbb{N}^*$ such that there exist $V_1, \dots V_n$ open Hausdorff and K quasi-compact satisfying $f^{-1}(\mathbb{C}^*) \subset K \subset V_1 \cup \dots \cup V_n$. For $n = 1$, for every $x \in V_1$, let $\mathcal{F}$ be a ultrafilter convergent to x. By Proposition 3.8, $\tilde{\mathcal{F}}$ is convergent; let S be its limit, then $\lim_{\mathcal{F}} f = \sum_{s \in S} f(s) = f(x)$, thus $f_{|V_1}$ is continuous.
Now assume the implication is true for $n-1$ ($n \geq 2$) and let us prove it for n. Since K is quasi-compact, there exist $V'_1, \dots, V'_n$ open sets, $K_1 \dots, K_n$ compact such that $K \subset V'_1 \cup \dots \cup V'_n$ and $V'_i \subset K_i \subset V_i$. Let $F = (V'_1 \cup \dots \cup V'_n) - (V'_1 \cup \dots \cup V'_{n-1})$. Then F is closed in V'_n and $f_{|F}$ is continuous. Moreover, $f_{|F} = 0$ outside $K' = K - (V'_1 \cup \dots \cup V'_{n-1})$ which is closed in K, hence quasi-compact, and Hausdorff, since $K' \subset V'_n$. Therefore, $f_{|F} \in C_c(F)$. It follows that there exists an extension $h \in C_c(V'_n)$ of $f_{|F}$. By considering $f - h$, we

may assume that $f = 0$ on F, so $f = 0$ outside $K' = K_1 \cup \cdots \cup K_{n-1}$. But $K' \subset V_1 \cup \cdots \cup V_{n-1}$, hence by induction hypothesis, $f \in C_c(X)$. □

COROLLARY 4.2. *Let X be a locally compact space, $f\colon X \to \mathbb{C}$, $f_n \in C_c(X)$. Suppose that there exists fixed quasi-compact set $Q \subset X$ such that $f_n^{-1}(\mathbb{C}^*) \subset Q$ for all n, and f_n converges uniformly to f. Then $f \in C_c(X)$.*

LEMMA 4.3. *Let X be a locally compact space. Let $(U_i)_{i\in I}$ be an open cover of X by Hausdorff subspaces. Then every $f \in C_c(X)$ is a finite sum $f = \sum f_i$, where $f_i \in C_c(U_i)$.*

Proof. See [6, Lemma 1.3]. □

LEMMA 4.4. *Let X and Y be locally compact spaces. Let $f \in C_c(X \times Y)$. Let V and W be open subspaces of X and Y such that $f^{-1}(\mathbb{C}^*) \subset Q \subset V \times W$ for some quasi-compact set Q. Then there exists a sequence $f_n \in C_c(V) \otimes C_c(W)$ such that $\lim_{n\to\infty} \|f - f_n\|_\infty = 0$.*

Proof. We may assume that $X = V$ and $Y = W$. Let (U_i) (resp. (V_j)) be an open cover of X (resp. Y) by Hausdorff subspaces. Then every element of $C_c(X\times Y)$ is a linear combination of elements of $C_c(U_i \times V_j)$ (Lemma 4.3). The conclusion follows from the fact that the image of $C_c(U_i)\otimes C_c(V_j) \to C_c(U_i\times V_j)$ is dense. □

LEMMA 4.5. *Let X be a locally compact space and $Y \subset X$ a closed subspace. Then the restriction map $C_c(X) \to C_c(Y)$ is well-defined and surjective.*

Proof. Let $(U_i)_{i\in I}$ be a cover of X by Hausdorff open subspaces. The map $C_c(U_i) \to C_c(U_i \cap Y)$ is surjective (since Y is closed), and $\oplus_{i\in I} C_c(U_i \cap Y) \to C_c(Y)$ is surjective (Lemma 4.3). Therefore, the map $\oplus_{i\in I} C_c(U_i) \to C_c(Y)$ is surjective. Since it is also the composition of the surjective map $\oplus_{i\in I} C_c(U_i) \to C_c(X)$ and of the restriction map $C_c(X) \to C_c(Y)$, the conclusion follows. □

4.2. HAAR SYSTEMS. Let G be a locally compact proper groupoid with Haar system (see definition below) such that $G^{(0)}$ is Hausdorff. If G is Hausdorff, then $C_c(G^{(0)})$ is endowed with the $C^*_r(G)$-valued scalar product $\langle \xi, \eta\rangle(g) = \overline{\xi(r(g))}\eta(s(g))$. Its completion is a $C^*_r(G)$-Hilbert module. However, if G is not Hausdorff, the function $g \mapsto \overline{\xi(r(g))}\eta(s(g))$ does not necessarily belong to $C_c(G)$, therefore we need a different construction in order to obtain a $C^*_r(G)$-module.

DEFINITION 4.6. [16, pp. 16-17] *Let G be a locally compact groupoid such that G^x is Hausdorff for every $x \in G^{(0)}$. A Haar system is a family of positive measures $\lambda = \{\lambda^x |\, x \in G^{(0)}\}$ such that $\forall x, y \in G^{(0)}$, $\forall \varphi \in C_c(G)$,*

(i) $\operatorname{supp}(\lambda^x) = G^x$*;*
(ii) $\lambda(\varphi)\colon x \mapsto \int_{g\in G^x} \varphi(g)\, \lambda^x(\mathrm{d}g) \quad \in C_c(G^{(0)})$;
(iii) $\int_{h\in G^x} \varphi(gh)\, \lambda^x(\mathrm{d}h) = \int_{h\in G^y} \varphi(h)\, \lambda^y(\mathrm{d}h)$.

Note that G^x is automatically Hausdorff if $G^{(0)}$ is Hausdorff (Prop. 2.8). Recall also [15, p. 36] that the range map for G is open.

LEMMA 4.7. *Let G be a locally compact groupoid with Haar system. Then for every quasi-compact subspace K of G, $\sup_{x\in G^{(0)}} \lambda^x(K\cap G^x)<\infty$.*

Proof. It is easy to show that there exists $f\in C_c(G)$ such that $1_K\le f$. Since $\sup_{x\in G^{(0)}}\lambda(f)(x)<\infty$, the conclusion follows. □

LEMMA 4.8. *Let G be a locally compact groupoid with Haar system such that $G^{(0)}$ is Hausdorff. Suppose that Z is a locally compact space and that $p\colon Z\to G^{(0)}$ is continuous. Then for every $f\in C_c(Z\times_{p,r}G)$, $\lambda(f)\colon z\mapsto \int_{g\in G^{p(z)}} f(z,g)\,\lambda^{p(z)}(\mathrm{d}g)$ belongs to $C_c(Z)$.*

Proof. By Lemma 4.5, f is the restriction of an element of $C_c(Z\times G)$.
If $f(z,g)=f_1(z)f_2(g)$, then $\psi(x)=\int_{g\in G^x} f_2(g)\,\lambda^x(dg)$ belongs to $C_c(G^{(0)})$, therefore $\psi\circ p\in C_b(Z)$. It follows that $\lambda(f)=f_1(\psi\circ p)$ belongs to $C_c(Z)$.
By linearity, if $f\in C_c(Z)\otimes C_c(G)$, then $\lambda(f)\in C_c(Z)$.
Now, for every $f\in C_c(Z\times G)$, there exist relatively quasi-compact open subspaces V and W of Z and G and a sequence $f_n\in C_c(V)\otimes C_c(W)$ such that f_n converges uniformly to f. From Lemma 4.7, $\lambda(f_n)$ converges uniformly to $\lambda(f)$, and $\lambda(f_n)\in C_c(Z)$. From Corollary 4.2, $\lambda(f)\in C_c(Z)$. □

PROPOSITION 4.9. *Let G be a locally compact groupoid with Haar system such that $G^{(0)}$ is Hausdorff. If G acts on a locally compact space Z with momentum map $p\colon Z\to G^{(0)}$, then $(\lambda^{p(z)})_{z\in Z}$ is a Haar system on $Z\rtimes G$.*

Proof. Results immediately from Lemma 4.8. □

5. THE HILBERT MODULE OF A PROPER GROUPOID

5.1. THE SPACE X'. Before we construct a Hilbert module associated to a proper groupoid, we need some preliminaries. Let G be a locally compact groupoid such that $G^{(0)}$ is Hausdorff. Denote by X' the closure of $G^{(0)}$ in $\mathcal{H}G$.

LEMMA 5.1. *Let G be a locally compact groupoid such that $G^{(0)}$ is Hausdorff. Then for all $S\in X'$, S is a subgroup of G.*

Proof. Since r and $s\colon G\to G^{(0)}$ extend continuously to maps $\mathcal{H}G\to G^{(0)}$, and since $r=s$ on $G^{(0)}$, one has $\mathcal{H}r=\mathcal{H}s$ on X', i.e. $\exists x_0\in G^{(0)}$, $S\subset G_{x_0}^{x_0}$.
Let $\mathcal{F}$ be a filter on $G^{(0)}$ whose limit is S. Then $a\in S$ if and only if a is a limit point of $\mathcal{F}$. Since for every $x\in G^{(0)}$ we have $x^{-1}x=x$, it follows that for every a, $b\in S$ one has $a^{-1}b\in S$, whence S is a subgroup of $G_{x_0}^{x_0}$. □

Denote by $q\colon X'\to G^{(0)}$ the map such that $S\subset G_{q(S)}^{q(S)}$. The map q is continuous since it is the restriction to X' of $\mathcal{H}r$.

LEMMA 5.2. *Let G be a locally compact proper groupoid such that $G^{(0)}$ is Hausdorff. Let $\mathcal{F}$ be a filter on X', convergent to S. Suppose that $q(\mathcal{F})$ converges to $S_0\in X'$. Then S_0 is a normal subgroup of S, and there exists $\Omega\in\mathcal{F}$ such that $\forall S'\in\Omega$, S' is group-isomorphic to S/S_0. In particular, $\{S'\in X' |\ \#S=\#S_0\#S'\}\in\mathcal{F}$.*

Proof. Using Proposition 3.12, we see that S is finite.
We shall use the notation $\tilde{\Omega}_{(V_i)_{i\in I}} = \Omega_{(V_i)_{i\in I}} \cap \Omega'_{\cup_{i\in I} V_i}$. Let $V'_s \subset V_s$ ($s \in S$) be Hausdorff, open neighborhoods of s, chosen small enough so that for some $\Omega \in \mathcal{F}$,

(a) $\Omega \subset \tilde{\Omega}_{(V'_s)_{s\in S}}$;
(b) $V'_{s_1} V'_{s_2} \subset V_{s_1 s_2}$, $\forall s_1,\, s_2 \in S$.
(c) $\forall s \in S - S_0$, $\forall S' \in \Omega$, $q(S') \notin V_s$;
(d) $q(\Omega) \subset \tilde{\Omega}_{(V_s)_{s\in S_0}}$;

Let $S' \in \Omega$. Let $\varphi\colon S \to S'$ such that $\{\varphi(s)\} = S' \cap V'_s$. Then φ is well-defined since $S' \cap V'_s \neq \emptyset$ (see (a)) and V'_s is Hausdorff.
If $s_1,\, s_2 \in S$ then $\varphi(s_i) \in S' \cap V'_{s_i}$. By (b), $\varphi(s_1)\varphi(s_2) \in S' \cap V_{s_1 s_2}$. Since $V_{s_1 s_2}$ is Hausdorff and also contains $\varphi(s_1 s_2) \in S'$, we have $\varphi(s_1 s_2) = \varphi(s_1)\varphi(s_2)$. This shows that φ is a group morphism.
The map φ is surjective, since $S' \subset \cup_{s\in S} V'_s$ (see (a)).
By (c), $\ker(\varphi) \subset S_0$ and by (d), $S_0 \subset \ker(\varphi)$. $\square$

Suppose now that the range map $r\colon G \to G^{(0)}$ is open. Then X' is endowed with an action of G (Prop. 3.10) defined by $S \cdot g = g^{-1}Sg = \{g^{-1}sg |\ s \in S\}$.

5.2. CONSTRUCTION OF THE HILBERT MODULE. Now, let G be a locally compact, proper groupoid. Assume that G is endowed with a Haar system, and that $G^{(0)}$ is Hausdorff. Let

$$\mathcal{E}^0 = \{f \in C_c(X') |\ f(S) = \sqrt{\#S} f(q(S))\ \forall S \in X'\}.$$

($q(S) \in G^{(0)}$ is identified to $\{q(S)\} \in X'$.)
Define, for all $\xi,\, \eta \in \mathcal{E}^0$ and $f \in C_c(G)$: $\langle \xi, \eta\rangle(g) = \overline{\xi(r(g))}\eta(s(g))$ and

$$(\xi f)(S) = \int_{g\in G^{q(S)}} \xi(g^{-1}Sg) f(g^{-1})\, \lambda^x(dg).$$

PROPOSITION 5.3. *With the above assumptions, the completion $\mathcal{E}(G)$ of $\mathcal{E}^0$ with respect to the norm $\|\xi\| = \|\langle \xi, \xi\rangle\|^{1/2}$ is a $C^*_r(G)$-Hilbert module.*

We won't give the direct proof here since this is a particular case of Theorem 7.8 (see Example 7.7(c)).

6. CUTOFF FUNCTIONS

If G is a locally compact Hausdorff proper groupoid with Haar system. Assume for simplicity that $G^{(0)}/G$ is compact. Then there exists a so-called "cutoff" function $c \in C_c(G^{(0)})_+$ such that for every $x \in G^{(0)}$, $\int_{g\in G^x} c(s(g))\, \lambda^x(\mathrm{d}g) = 1$, and the function $g \mapsto \sqrt{c(r(g))c(s(g))}$ defines projection in $C^*_r(G)$. However, if G is not Hausdorff, then the above function does not belong to $C_c(G)$ is general, thus we need another definition of a cutoff function.

Let $X'_{\geq k} = \{S \in X' |\ \#S \geq k\}$. By Lemma 3.11, $X'_{\geq k}$ is closed.

LEMMA 6.1. *Let G be a locally compact, proper groupoid with $G^{(0)}$ Hausdorff. Let $X_{\geq k} = q(X'_{\geq k})$. Then $X_{\geq k}$ is closed in $G^{(0)}$.*

Proof. It suffices to show that for every compact subspace K of $G^{(0)}$, $X_{\geq k} \cap K$ is closed. Let $K' = G_K^K$. Then K' is quasi-compact, and from Proposition 3.7, $K'' = \{S \in \mathcal{H}G |\ S \cap K' \neq \emptyset\}$ is compact. The set $q^{-1}(K) \cap X'_{\geq k} = K'' \cap X'_{\geq k}$ is closed in K'', hence compact; its image by q is $X_{\geq k} \cap K$. □

LEMMA 6.2. *Let G be a locally compact, proper groupoid, with $G^{(0)}$ Hausdorff. Let $\alpha \in \mathbb{R}$. For every compact set $K \subset G^{(0)}$, there exists $f\colon X'_K \to \mathbb{R}_+^*$ continuous, where $X'_K = q^{-1}(K) \subset X'$, such that*

$$\forall S \in X'_K, \quad f(S) = f(q(S))(\#S)^\alpha.$$

Proof. Let $K' = G_K^K$. It is closed and quasi-compact. From Proposition 3.7, X'_K is quasi-compact. For every $S \in X'_K$, we have $S \subset K'$. By Proposition 3.12, there exists $n \in \mathbb{N}^*$ such that $X'_{\geq n+1} \cap X'_K = \emptyset$. We can thus proceed by reverse induction: suppose constructed $f_{k+1}\colon X'_K \cap q^{-1}(X_{\geq k+1}) \to \mathbb{R}_+^*$ continuous such that $f_{k+1}(S) = f_{k+1}(q(S))(\#S)^\alpha$ for all $S \in X'_K \cap q^{-1}(X_{\geq k+1})$.
Since $X'_K \cap q^{-1}(X_{\geq k+1})$ is closed in the compact set $X'_K \cap q^{-1}(X_{\geq k})$, there exists a continuous extension $h\colon X'_K \cap q^{-1}(X_{\geq k}) \to \mathbb{R}$ of f_{k+1}. Replacing $h(x)$ by $\sup(h(x), \inf f_{k+1})$, we may assume that $h(X'_K \cap q^{-1}(X_{\geq k})) \subset \mathbb{R}_+^*$. Put $f_k(S) = h(q(S))(\#S)^\alpha$. Let us show that f_k is continuous.
Let $\mathcal{F}$ be a ultrafilter on $X'_K \cap q^{-1}(X_{\geq k})$, and let S be its limit. Since $q(\mathcal{F})$ is a ultrafilter on K, it has a limit $S_0 \in X'_K$.
For every $S_1 \in q^{-1}(X_{\geq k})$, choose $\psi(S_1) \in X'_{\geq k}$ such that $q(S_1) = q(\psi(S_1))$. Let $S' \in X'_K \cap X'_{\geq k}$ be the limit of $\psi(\mathcal{F})$.
From Lemma 5.2, $\Omega_1 = \{S_1 \in X'_K \cap q^{-1}(X_{\geq k}) |\ \#S = \#S_0 \#S_1\}$ is an element of $\mathcal{F}$, and $\Omega_2 = \{S_2 \in X'_{\geq k} |\ \#S' = \#S_0 \#S_2\}$ is an element of $\psi(\mathcal{F})$.

- If $\#S_0 > 1$, then $S' \in X_{\geq k+1}$, so S and S_0 belong to $q^{-1}(X_{\geq k+1})$. Therefore, $f_k(S_1) = (\#S_1)^\alpha h(q(S_1))$ converges with respect to $\mathcal{F}$ to
$$\begin{aligned}\frac{(\#S)^\alpha}{(\#S_0)^\alpha} h(S_0) &= \frac{(\#S)^\alpha}{(\#S_0)^\alpha} f_{k+1}(S_0) = f_{k+1}(S) \\ &= f_{k+1}(q(S))(\#S)^\alpha = h(q(S))(\#S)^\alpha = f_k(S).\end{aligned}$$
- If $S_0 = \{q(S)\}$, then $f_k(S_1) = (\#S_1)^\alpha h(q(S_1))$ converges with respect to $\mathcal{F}$ to $(\#S)^\alpha h(q(S)) = f_k(S)$.

Therefore, f_k is a continuous extension of f_{k+1}. □

THEOREM 6.3. *Let G be a locally compact, proper groupoid such that $G^{(0)}$ is Hausdorff and $G^{(0)}/G$ is σ-compact. Let $\pi\colon G^{(0)} \to G^{(0)}/G$ be the canonical mapping. Then there exists $c\colon X' \to \mathbb{R}_+$ continuous such that*

(a) *$c(S) = c(q(S))\#S$ for all $S \in X'$;*
(b) *$\forall \alpha \in G^{(0)}/G$, $\exists x \in \pi^{-1}(\alpha)$, $c(x) \neq 0$;*
(c) *$\forall K \subset G^{(0)}$ compact, $\mathrm{supp}(c) \cap q^{-1}(F)$ is compact, where $F = s(G^K)$.*

If moreover G admits a Haar system, then there exists $c\colon X' \to \mathbb{R}_+$ continuous satisfying (a), (b), (c) and

(d) $\forall x \in G^{(0)}, \quad \displaystyle\int_{g \in G^x} c(s(g))\, \lambda^x(dg) = 1.$

Proof. There exists a locally finite cover (V_i) of $G^{(0)}/G$ by relatively compact open subspaces. Since π is open and $G^{(0)}$ is locally compact, there exists $K_i \subset G^{(0)}$ compact such that $\pi(K_i) \supset V_i$. Let (φ_i) be a partition of unity associated to the cover (V_i). For every i, from Lemma 6.2, there exists $c_i\colon X'_{K_i} \to \mathbb{R}_+^*$ continuous such that $c_i(S) = c_i(q(S))\#S$ for all $S \in X'_{K_i}$. Let

$$c(S) = \sum_i c_i(S)\varphi_i(\pi(q(S))).$$

It is clear that c is continuous from X' to $\mathbb{R}_+$, and that $c(S) = c(q(S))\#S$.
Let us prove (b): let $x_0 \in G^{(0)}$. There exists i such that $\varphi_i(\pi(x_0)) \neq 0$. Choose $x \in K_i$ such that $\pi(x) = \pi(x_0)$, then $c(x) \geq c_i(x)\varphi_i(\pi(x_0)) > 0$.
Let us show (c). Note that $F = \pi^{-1}(\pi(K))$ is closed, so $q^{-1}(F)$ is closed. Let K_1 be a compact neighborhood of K and $F_1 = \pi^{-1}(\pi(K_1))$. Let $J = \{i|\ V_i \cap \pi(K_1) \neq \emptyset\}$. Then for all $i \notin J$, $c_i(\varphi_i \circ \pi \circ q) = 0$ on $q^{-1}(F_1)$, therefore $c = \sum_{j\in J} c_j(\varphi_j \circ \pi \circ q)$ in a neighborhood of $q^{-1}(F)$. Since for all i, $\operatorname{supp}(c_i(\varphi_i \circ \pi \circ q))$ is compact and since J is finite, $\operatorname{supp}(c) \cap q^{-1}(F) \subset \cup_{i\in J}\operatorname{supp}(c_i(\varphi_i \circ \pi \circ q))$ is compact.
Let us show the last assertion. Let $\varphi(g) = c(s(g))$. Let $\mathcal{F}$ be a filter on G convergent in $\mathcal{H}G$ to $A \subset G$. Choose $a \in A$ and let $S = a^{-1}A$. Then $s(\mathcal{F})$ converges to S in $\mathcal{H}G$, hence

$$\lim_{\mathcal{F}} \varphi = \#Sc(s(a)) = \sum_{g\in S} c(s(g)) = \sum_{g\in S} \varphi(g).$$

For every compact set $K \subset G^{(0)}$,

$$\begin{aligned}
&\{g \in G|\ r(g) \in K \text{ and } \varphi(g) \neq 0\}\\
&\quad \subset \ \{g \in G|\ r(g) \in K \text{ and } s(g) \in \operatorname{supp}(c)\}\\
&\quad \subset \ G^K_{q(\operatorname{supp}(c)\cap q^{-1}(F))},
\end{aligned}$$

so $G^K \cap \{g \in G|\ \varphi(g) \neq 0\}$ is included in a quasi-compact set. Therefore, for every $l \in C_c(G^{(0)})$, $g \mapsto l(r(g))\varphi(g)$ belongs to $C_c(G)$. It follows that $h(x) = \int_{g\in G^x} \varphi(g)\,\lambda^x(dg)$ is a continuous function. Moreover, for every $x \in G^{(0)}$ there exists $g \in G^x$ such that $\varphi(g) \neq 0$, so $h(x) > 0\ \forall x \in G^{(0)}$. It thus suffices to replace $c(x)$ by $c(x)/h(x)$. $\square$

EXAMPLE 6.4. *In Example 2.3 with $\Gamma = \mathbb{Z}_n$ and $H = \{0\}$, the cutoff function is the unique continuous extension to X' of the function $c(x) = 1$ for $x \in (0, 1]$, and $c(0) = 1/n$.*

PROPOSITION 6.5. *Let G be a locally compact, proper groupoid with Haar system such that $G^{(0)}$ is Hausdorff and $G^{(0)}/G$ is compact. Let c be a cutoff function. Then the function $p(g) = \sqrt{c(r(g))c(s(g))}$ defines a selfadjoint projection $p \in C_r^*(G)$, and $\mathcal{E}(G)$ is isomorphic to $pC_r^*(G)$.*

Proof. Let $\xi_0(x) = \sqrt{c(x)}$. Then one easily checks that $\xi_0 \in \mathcal{E}^0$, $\langle \xi_0, \xi_0 \rangle = p$ and $\xi_0 \langle \xi_0, \xi_0 \rangle = \xi_0$, therefore p is a selfadjoint projection in $C_r^*(G)$. The maps

$$\begin{aligned} &\mathcal{E}(G) \to pC_r^*(G), && \xi \mapsto \langle \xi_0, \xi \rangle = p \langle \xi_0, \xi \rangle \\ &pC_r^*(G) \to \mathcal{E}(G), && a \mapsto \xi_0 a = \xi_0 p a \end{aligned}$$

are inverses from each other. □

7. Generalized morphisms and C^*-algebra correspondences

UNTIL THE END OF THE PAPER, ALL GROUPOIDS ARE ASSUMED LOCALLY COMPACT, WITH OPEN RANGE MAP. In this section, we introduce a notion of generalized morphism for locally compact groupoids which are not necessarily Hausdorff, and a notion of locally proper generalized morphism.
Then, we show that a locally proper generalized morphism from G_1 to G_2 which satisfies an additional condition induces a $C_r^*(G_1)$-module $\mathcal{E}$ and a $*$-morphism $C_r^*(G_2) \to \mathcal{K}(\mathcal{E})$, hence an element of $KK(C_r^*(G_2), C_r^*(G_1))$.

7.1. Generalized morphisms.

DEFINITION 7.1. [4, 5, 8, 9, 12, 14] *Let G_1 and G_2 be two groupoids. A generalized morphism from G_1 to G_2 is a triple (Z, ρ, σ) where*

$$G_1^{(0)} \xleftarrow{\rho} Z \xrightarrow{\sigma} G_2^{(0)},$$

Z is endowed with a left action of G_1 with momentum map ρ and a right action of G_2 with momentum map σ which commute, such that

(a) *the action of G_2 is free and ρ-proper,*
(b) *ρ induces a homeomorphism $Z/G_2 \simeq G_1^{(0)}$.*

In Definition 7.1, one may replace (b) by (b)' or (b)" below:

(b)' ρ is open and induces a bijection $Z/G_2 \to G_1^{(0)}$.
(b)" the map $Z \rtimes G_2 \to Z \times_{G_1^{(0)}} Z$ defined by $(z, \gamma) \mapsto (z, z\gamma)$ is a homeomorphism.

EXAMPLE 7.2. *Let G_1 and G_2 be two groupoids.If $f \colon G_1 \to G_2$ is a groupoid morphism, let $Z = G_1^{(0)} \times_{f,r} G_2$, $\rho(x, \gamma) = x$ and $\sigma(x, \gamma) = s(\gamma)$. Define the actions of G_1 and G_2 by $g \cdot (x, \gamma) \cdot \gamma' = (r(g), f(g)\gamma\gamma')$. Then (Z, ρ, σ) is a generalized morphism from G_1 to G_2.*

That ρ is open follows from the fact that the range map $G_2 \to G_2^{(0)}$ is open and from Lemma 2.25. The other properties in Definition 7.1 are easy to check.

7.2. Locally proper generalized morphisms.

DEFINITION 7.3. *Let G_1 and G_2 be two groupoidsA generalized morphism from G_1 to G_2 is said to be* locally proper *if the action of G_1 on Z is σ-proper.*

Our terminology is justified by the following proposition:

PROPOSITION 7.4. *Let G_1 and G_2 be two groupoids such that $G_2^{(0)}$ is Hausdorff. Let $f\colon G_1 \to G_2$ be a groupoid morphism. Then the associated generalized groupoid morphism is locally proper if and only if the map $(f, r, s)\colon G_1 \to G_2 \times G_1^{(0)} \times G_1^{(0)}$ is proper.*

Proof. Let $\varphi\colon G_1 \times_{f\circ s,r} G_2 \to (G_2 \times_{s,s} G_2) \times_{r\times r, f\times f} (G_1^{(0)} \times G_1^{(0)})$ defined by $\varphi(g_1, g_2) = (f(g_1)g_2, g_2, r(g_1), s(g_1))$. By definition, the action of G_1 on Z is proper if and only if φ is a proper map. Consider $\theta\colon G_2 \times_{s,s} G_2 \to G_2^{(2)}$ given by $(\gamma, \gamma') = (\gamma(\gamma')^{-1}, \gamma')$. Let $\psi = (\theta \times 1) \circ \varphi$. Since θ is a homeomorphism, the action of G_1 on Z is proper if and only if ψ is proper.
Suppose that (f, r, s) is proper. Let $f' = (f, r, s) \times 1\colon G_1 \times G_2 \to G_2 \times G_1^{(0)} \times G_1^{(0)} \times G_2$. Then f' is proper. Let $F = \{(\gamma, x, x', \gamma') \in G_2 \times G_1^{(0)} \times G_1^{(0)} \times G_2 |\ s(\gamma) = r(\gamma') = f(x'),\ r(\gamma) = f(x)\}$. Then $f'\colon (f')^{-1}(F) \to F$ is proper, i.e. ψ is proper.
Conversely, suppose that ψ is proper. Let $F' = \{(\gamma, y, x, x') \in G_2 \times G_2^{(0)} \times G_1^{(0)} \times G_1^{(0)} |\ s(\gamma) = y\}$. Then $\psi\colon \psi^{-1}(F') \to F'$ is proper, therefore (f, r, s) is proper. □

Our objective is now to show the

PROPOSITION 7.5. *Let G_1, G_2, G_3 be groupoidsLet (Z_1, ρ_1, σ_1) and (Z_2, ρ_2, σ_2) be two generalized groupoid morphisms from G_1 to G_2 and from G_2 to G_3 respectively. Then $(Z, \rho, \sigma) = (Z_1 \times_{G_2} Z_2, \rho_1 \times 1, 1 \times \sigma_2)$ is a generalized groupoid morphism. If (Z_1, ρ_1, σ_1) and (Z_2, ρ_2, σ_2) are locally proper, then (Z, ρ, σ) is locally proper.*

Proposition 7.5 shows that groupoids form a category whose arrows are generalized morphisms, and that two groupoids are isomorphic in that category if and only if they are Morita-equivalent. Moreover, the same conclusions hold for the category whose arrows are locally proper generalized morphisms. In particular, local properness of generalized morphisms is invariant under Morita-equivalence.
All the assertions of Proposition 7.5 follow from Lemma 2.33.

7.3. Proper generalized morphisms.

DEFINITION 7.6. *Let G_1 and G_2 be groupoids. A generalized morphism (Z, ρ, σ) from G_1 to G_2 is said to be* proper *if it is locally proper, and if for every quasi-compact subspace K of $G_2^{(0)}$, $\sigma^{-1}(K)$ is G_1-compact.*

EXAMPLES 7.7.
(a) *Let X and Y be locally compact spaces and $f\colon X \to Y$ a continuous map. Then the generalized morphism (X, Id, f) is proper if and only if f is proper.*
(b) *Let $f\colon G_1 \to G_2$ be a continuous morphism between two locally compact groups. Let $p\colon G_2 \to \{*\}$. Then (G_2, p, p) is proper if and only if f is proper and $f(G_1)$ is co-compact in G_2.*

(c) *Let G be a locally compact proper groupoid with Haar system such that $G^{(0)}$ is Hausdorff, and let $\pi\colon G^{(0)} \to G^{(0)}/G$ be the canonical mapping. Then $(G^{(0)}, \mathrm{Id}, \pi)$ is a proper generalized morphism from G to $G^{(0)}/G$.*

7.4. CONSTRUCTION OF A C^*-CORRESPONDENCE. Until the end of the section, our goal is to prove:

THEOREM 7.8. *Let G_1 and G_2 be locally compact groupoids with Haar system such that $G_1^{(0)}$ and $G_2^{(0)}$ are Hausdorff, and (Z, ρ, σ) a locally proper generalized morphism from G_1 to G_2. Then one can construct a $C_r^*(G_1)$-Hilbert module $\mathcal{E}_Z$ and a map $\pi\colon C_r^*(G_2) \to \mathcal{L}(\mathcal{E}_Z)$. Moreover, if (Z, ρ, σ) is proper, then π maps to $\mathcal{K}(\mathcal{E}_Z)$. Therefore, it gives an element of $KK(C_r^*(G_2), C_r^*(G_1))$.*

COROLLARY 7.9. *(see* [14]*) Let G_1 and G_2 be locally compact groupoids with Haar system such that $G_1^{(0)}$ and $G_2^{(0)}$ are Hausdorff. If G_1 and G_2 are Morita-equivalent, then $C_r^*(G_1)$ and $C_r^*(G_2)$ are Morita-equivalent.*

COROLLARY 7.10. *Let $f\colon G_1 \to G_2$ be morphism between two locally compact groupoids with Haar system such that $G_1^{(0)}$ and $G_2^{(0)}$ are Hausdorff. If the restriction of f to $(G_1)_K^K$ is proper for each compact set $K \subset (G_1)^{(0)}$ then f induces a correspondence $\mathcal{E}_f$ from $C_r^*(G_2)$ to $C_r^*(G_1)$. If in addition for every compact set $K \subset G_2^{(0)}$ the quotient of $G_1^{(0)} \times_{f,r} (G_2)_K$ by the diagonal action of G_1 is compact, then $C_r^*(G_2)$ maps to $\mathcal{K}(\mathcal{E}_f)$ and thus f defines a KK-element $[f] \in KK(C_r^*(G_2), C_r^*(G_1))$.*

Proof. See Proposition 7.4 and Definition 7.6 applied to the generalized morphism $Z_f = G_1^{(0)} \times_{f,r} G_2$ as in Example 7.2 □

The rest of the section is devoted to proving Theorem 7.8.
Let us first recall the construction of the correspondence when the groupoids are Hausdorff [11]. It is the closure of $C_c(Z)$ with the $C_r^*(G_1)$-valued scalar product

$$\langle \xi, \eta \rangle(g) = \int_{\gamma \in (G_2)^{\sigma(z)}} \overline{\xi(z\gamma)}\eta(g^{-1}z\gamma)\, \lambda^{\sigma(z)}(\mathrm{d}\gamma), \tag{2}$$

where z is an arbitrary element of Z such that $\rho(z) = r(g)$. The right $C_r^*(G_1)$-module structure is defined $\forall \xi \in C_c(Z)$, $\forall a \in C_c(G_1)$ by

$$(\xi a)(z) = \int_{g \in (G_1)^{\rho(z)}} \xi(g^{-1}z)a(g^{-1})\, \lambda^{\rho(z)}(\mathrm{d}g), \tag{3}$$

and the left action of $C_r^*(G_2)$ is

$$(b\xi)(z) = \int_{\gamma \in (G_2)^{\sigma(z)}} b(\gamma)\xi(z\gamma)\, \lambda^{\sigma(z)}(\mathrm{d}\gamma) \tag{4}$$

for all $b \in C_c(G_2)$.

We now come back to non-Hausdorff groupoids. For every open Hausdorff set $V \subset Z$, denote by V' its closure in $\mathcal{H}((G_1 \ltimes Z)_V^V)$, where $z \in V$ is identified

to $(\rho(z), z) \in \mathcal{H}((G_1 \ltimes Z)_V^V)$. Let $\mathcal{E}_V^0$ be the set of $\xi \in C_c(V')$ such that $\xi(z) = \dfrac{\xi(S \times \{z\})}{\sqrt{\#S}}$ for all $S \times \{z\} \in V'$.

LEMMA 7.11. *The space $\mathcal{E}_Z^0 = \sum_{i \in I} \mathcal{E}_{V_i}^0$ is independent of the choice of the cover (V_i) of Z by Hausdorff open subspaces.*

Proof. It suffices to show that for every open Hausdorff subspace V of Z, one has $\mathcal{E}_V^0 \subset \sum_{i \in I} \mathcal{E}_{V_i}^0$. Let $\xi \in \mathcal{E}_V^0$. Denote by $q_V : V' \to V$ the canonical map defined by $q_V(S \times \{z\}) = z$. Let $K \subset V$ compact such that $\mathrm{supp}(\xi) \subset q_V^{-1}(K)$. There exists $J \subset I$ finite such that $K \subset \cup_{j \in J} V_j$. Let $(\varphi_j)_{j \in J}$ be a partition of unity associated to that cover, and $\xi_j = \xi.(\varphi_j \circ q_V)$. One easily checks that $\xi_j \in \mathcal{E}_{V_j}^0$ and that $\xi = \sum_{j \in J} \xi_j$. □

We now define a $C_r^*(G_1)$-valued scalar product on $\mathcal{E}_Z^0$ by Eqn. (2) where z is an arbitrary element of Z such that $\rho(z) = r(g)$. Our definition is independent of the choice of z, since if z' is another element, there exists $\gamma' \in G_2$ such that $z' = z\gamma'$, and the Haar system on G_2 is left-invariant.
Moreover, the integral is convergent for all $g \in G_1$ because the action of G_2 on Z is proper.
Let us show that $\langle \xi, \eta \rangle \in C_c(G_1)$ for all ξ, $\eta \in \mathcal{E}_Z^0$. We need a preliminary lemma:

LEMMA 7.12. *Let X and Y be two topological spaces such that X is locally compact and $f : X \to Y$ proper. Let $\mathcal{F}$ be a ultrafilter such that f converges to $y \in Y$ with respect to $\mathcal{F}$. Then there exists $x \in X$ such that $f(x) = y$ and $\mathcal{F}$ converges to x.*

Proof. Let $Q = f^{-1}(y)$. Since f is proper, Q is quasi-compact. Suppose that for all $x \in Q$, $\mathcal{F}$ does not converge to x. Then there exists an open neighborhood V_x of x such that $V_x^c \in \mathcal{F}$. Extracting a finite cover $(V_1, \dots, V_n)$ of Q, there exists an open neighborhood V of Q such that $V^c \in \mathcal{F}$. Since f is closed, $f(V^c)^c$ is a neighborhood of y. By assumption, $f(V^c)^c \in f(\mathcal{F})$, i.e. $\exists A \in \mathcal{F}$, $f(A) \subset f(V^c)^c$. This implies that $A \subset V$, therefore $V \in \mathcal{F}$: this contradicts $V^c \in \mathcal{F}$.
Consequently, there exists $x \in Q$ such that $\mathcal{F}$ converges to x. □

To show that $\langle \xi, \eta \rangle \in C_c(G_1)$, we can suppose that $\xi \in \mathcal{E}_U^0$ and $\eta \in \mathcal{E}_V^0$, where U and V are open Hausdorff. Let $F(g, z) = \overline{\xi(z)}\eta(g^{-1}z)$, defined on $\Gamma = G_1 \times_{r,\rho} Z$. Since the action of G_1 on Z is proper, F is quasi-compactly supported. Let us show that $F \in C_c(\Gamma)$.
Let $\mathcal{F}$ be a ultrafilter on Γ, convergent in $\mathcal{H}\Gamma$. Since $G_1^{(0)}$ is Hausdorff, its limit has the form $S = S'g_0 \times S''$ where $S' \subset (G_1)_{r(g_0)}^{r(g_0)}$, $S'' \subset \rho^{-1}(r(g_0))$. Moreover, S' is a subgroup of $(G_1)_{r(g)}^{r(g)}$ by the proof of Lemma 5.1.
Suppose that there exist $z_0, z_1 \in S''$ and $g_1 \in S'g_0$ such that $z_0 \in U$ and $g_1^{-1}z_1 \in V$. By Lemma 7.12 applied to the proper map $G_1 \rtimes Z \to Z \times Z$, there exists $s_0 \in S'$ such that $z_0 = s_0 z_1$. We may assume that $g_0 = s_0 g_1$. Then

$\sum_{s\in S} F(s) = \sum_{s'\in S'} \overline{\xi(z_0)}\eta(g_0^{-1}(s')^{-1}z_0)$. If $s' \notin \mathrm{stab}(z_0)$, then $g_0^{-1}(s')^{-1}z_0 \notin V$ since $g_0^{-1}z_0$ and $g_0^{-1}(s')^{-1}z_0$ are distinct limits of $(g,z)\mapsto g^{-1}z$ with respect to $\mathcal{F}$ and V is Hausdorff. Therefore,

$$\begin{aligned}
\sum_{s\in S} F(s) &= \#(\mathrm{stab}(z_0)\cap S')\overline{\xi(z_0)}\eta(g_0^{-1}z_0)\\
&= \overline{\sqrt{\#(\mathrm{stab}(z_0)\cap S')}\xi(z_0)}\sqrt{\#(\mathrm{stab}(g_0^{-1}z_0)\cap(g_0^{-1}S'g_0))}\eta(z_0)\\
&= \lim_{\mathcal{F}} \overline{\xi(z)}\eta(g^{-1}z) = \lim_{\mathcal{F}} F(g,z).
\end{aligned}$$

If for all z_0, $z_1 \in S''$ and all $g_1 \in S'g_0$, $(z_0, g_1^{-1}z_1) \notin U\times V$, then $\sum_{s\in S} F(g,z) = 0 = \lim_{\mathcal{F}} F(g,z)$.
By Proposition 4.1, $F \in C_c(\Gamma)$.
Since $\langle\xi,\eta\rangle(g) = \int_{\gamma\in(G_2)^{\sigma(z)}} F(g,z\gamma)\,\lambda^{\sigma(z)}(\mathrm{d}\gamma)$, to prove that $\langle\xi,\eta\rangle \in C_c(G_1)$ it suffices to show:

LEMMA 7.13. *Let G_1 and G_2 be two locally compact groupoids with Haar system such that $G_i^{(0)}$ are Hausdorff. Let (Z,ρ,σ) be a generalized morphism from G_1 to G_2. Let $\Gamma = G_1 \times_{r,\rho} Z$. Then for every $F \in C_c(\Gamma)$, the function*

$$g \mapsto \int_{\gamma\in(G_2)^{\sigma(z)}} F(g,z\gamma)\,\lambda^{\sigma(z)}(\mathrm{d}\gamma),$$

where $z\in Z$ is an arbitrary element such that $\rho(z) = r(g)$, belongs to $C_c(G_1)$.

Proof. Suppose first that $F(g,z) = f(g)h(z)$, where $f \in C_c(G_1)$ and $h \in C_c(Z)$. Let $H(z) = \int_{\gamma\in(G_2)^{\sigma(z)}} h(z\gamma)\,\lambda^{\sigma(z)}(\mathrm{d}\gamma)$. By Lemma 7.14 below (applied to the groupoid $Z \rtimes G_2$), H is continuous. It is obviously G_2-invariant, therefore $H \in C_c(Z/G_2)$. Let $\tilde{H} \in C_c(G_1^{(0)}) \simeq C_c(Z/G_2)$ correspond to H. The map

$$g \mapsto \int_{\gamma\in(G_2)^{\sigma(z)}} F(g,z\gamma)\,\lambda^{\sigma(z)}(\mathrm{d}\gamma) = f(g)\tilde{H}(s(g))$$

thus belongs to $C_c(G_1)$.
By linearity, the lemma is true for $F \in C_c(G_1)\otimes C_c(Z)$. By Lemma 4.4 and Lemma 4.5, F is the uniform limit of functions $F_n \in C_c(G_1)\otimes C_c(Z)$ which are supported in a fixed quasi-compact set $Q = Q_1\times Q_2 \subset G_1\times Z$. Let $Q' \subset Z$ quasi-compact such that $\rho(Q') \supset r(Q_1)$. Since the action of G_2 on Z is proper, $K = \{\gamma\in G_2 \mid Q'\gamma\cap Q_2 \neq \emptyset\}$ is quasi-compact. Using the fact that $G_1^{(0)} \simeq Z/G_2$, it is easy to see that

$$\sup_{(g,z)\in\Gamma} \int_{\gamma\in(G_2)^{\sigma(z)}} 1_Q(g,z\gamma)\,\lambda^{\sigma(z)}(\mathrm{d}\gamma) \le \sup_{z\in Q'} \int_{\gamma\in G_2^{\sigma(z)}} 1_{Q_2}(z\gamma)\lambda^{\sigma(z)}(\mathrm{d}\gamma)$$

$$\le \sup_{x\in G_2^{(0)}} \int_{\gamma\in G_2^x} 1_K(\gamma)\lambda^x(\mathrm{d}\gamma) < \infty$$

by Lemma 4.7. Therefore,

$$\lim_{n\to\infty}\sup_{g\in G_1}\left|\int_{\gamma\in G_2^{\sigma(z)}} F(g,z\gamma)-F_n(g,z\gamma)\,\lambda^{\sigma(z)}(\mathrm{d}\gamma)\right|=0.$$

The conclusion follows from Corollary 4.2. □

In the proof of Lemma 7.13 we used the

LEMMA 7.14. *Let G be a locally compact, proper groupoid with Haar system, such that G^x is Hausdorff for all $x\in G^{(0)}$, and $G_x^x=\{x\}$ for all $x\in G^{(0)}$. We do not assume $G^{(0)}$ to be Hausdorff. Then $\forall f\in C_c(G^{(0)})$,*

$$\varphi\colon G^{(0)}\to\mathbb{C},\quad x\mapsto\int_{g\in G^x} f(s(g))\,\lambda^x(dg)$$

is continuous.

Proof. Let V be an open, Hausdorff subspace of $G^{(0)}$. Let $h\in C_c(V)$. Since $(r,s)\colon G\to G^{(0)}\times G^{(0)}$ is a homeomorphism from G onto a closed subspace of $G^{(0)}\times G^{(0)}$, and $(x,y)\mapsto h(x)f(y)$ belongs to $C_c(G^{(0)}\times G^{(0)})$, the map $g\mapsto h(r(g))f(s(g))$ belongs to $C_c(G)$, therefore by definition of a Haar system, $x\mapsto\int_{g\in G^x} h(r(g))f(s(g))\,\lambda^x(dg)=h(x)\varphi(x)$ belongs to $C_c(G^{(0)})$.
Since $h\in C_c(V)$ is arbitrary, this shows that $\varphi_{|V}$ is continuous, hence φ is continuous on $G^{(0)}$. □

Now, let us show the positivity of the scalar product. Recall that for all $x\in G_1^{(0)}$ there is a representation $\pi_{G_1,x}\colon C^*(G_1)\to\mathcal{L}(L^2(G_1^x))$ such that for all $a\in C_c(G_1)$ and all $\eta\in C_c(G_1^x)$,

$$(\pi_{G_1,x}(a)\eta)(g)=\int_{h\in G_1^{s(g)}} a(h)\eta(gh)\,\lambda^{s(g)}(\mathrm{d}h).$$

By definition, $\|a\|_{C_r^*(G_1)}=\sup_{x\in G_1^{(0)}}\|\pi_{G_1,x}(a)\|$.

$$\begin{aligned}\langle\eta,\pi_{G_1,x}(a)\eta\rangle &= \int_{g\in G_1^x,\ h\in G_1^{s(g)}}\overline{\eta(g)}a(h)\eta(gh)\,\lambda^{s(g)}(\mathrm{d}h)\lambda^x(dg)\\ &= \int_{g\in G_1^x,\ h\in G^{s(g)}}\overline{\eta(g)}a(g^{-1}h)\eta(h)\,\lambda^x(dg)\lambda^x(dh).\end{aligned}$$

Fix $z\in Z$ such that $\rho(z)=x$. Replacing $a(g^{-1}h)$ by

$$\langle\xi,\xi\rangle(g^{-1}h)=\int_{\gamma\in G_2^{\sigma(z)}}\overline{\xi(g^{-1}z\gamma)}\xi(h^{-1}z\gamma)\,\lambda^{\sigma(z)}(\mathrm{d}\gamma),$$

we get

$$\langle\eta,\pi_{G_1,x}(\langle\xi,\xi\rangle)\eta\rangle=\int_{\gamma\in G_2^{\sigma(z)}}\lambda^{\sigma(z)}(\mathrm{d}\gamma)\left|\int_{g\in G^x}\eta(g)\xi(g^{-1}z\gamma)\,\lambda^x(dg)\right|^2. \tag{5}$$

It follows that $\pi_{G_1,x}(\langle\xi,\xi\rangle)\geq 0$ for all $x\in G_1^{(0)}$, so $\langle\xi,\xi\rangle\geq 0$ in $C_r^*(G_1)$.

Now, let us define a $C_r^*(G_1)$-module structure on $\mathcal{E}_Z^0$ by Eqn.(3) for all $\xi \in \mathcal{E}_Z^0$ and $a \in C_c(G_1)$.
Let us show that $\xi a \in \mathcal{E}_Z^0$. We need a preliminary lemma:

LEMMA 7.15. *Let X and Y be quasi-compact spaces, (Ω_k) an open cover of $X \times Y$. Then there exist finite open covers (X_i) and (Y_j) of X and Y such that $\forall i, j\ \exists k$, $X_i \times Y_j \subset \Omega_k$.*

Proof. For all $(x, y) \in X \times Y$ choose open neighborhoods $U_{x,y}$ and $V_{x,y}$ of x and y such that $U_{x,y} \times V_{x,y} \subset \Omega_k$ for some k. For y fixed, there exist $x_1, \ldots, x_n$ such that $(U_{x_i,y})_{1\le i\le n}$ covers X. Let $V_y = \cap_{i=1}^n U_{x_i,y}$. Then for all $(x, y) \in X \times Y$, there exists an open neighborhood $U'_{x,y}$ of x and k such that $U'_{x,y} \times V_y \subset \Omega_k$. Let $(V_1, \ldots, V_m) = (V_{y_1}, \ldots, V_{y_m})$ such that $\cup_{1\le j\le m} V_j = Y$. For all $x \in X$, let $U'_x = \cap_{j=1}^m U'_{x,y_j}$. Let $(U_1, \ldots, U_p)$ be a finite sub-cover of $(U'_x)_{x\in X}$. Then for all i and for all j, there exists k such that $U_i \times V_j \subset \Omega_k$. $\square$

Let Q_1 and Q_2 be quasi-compact subspaces of G_1 of Z respectively such that $a^{-1}(\mathbb{C}^*) \subset Q_1$ and $\xi^{-1}(\mathbb{C}^*) \subset Q_2$. Let Q be a quasi-compact subspace of Z such that $\forall g \in Q_1$, $\forall z \in Q_2$, $g^{-1}z \in Q$. Let (U_k) be a finite cover of Q by Hausdorff open subspaces of Z. Let $Q' = Q_1 \times_{r,\rho} Q_2$. Then Q' is a closed subspace of $Q_1 \times Q_2$. Let $\Omega'_k = \{(g, z) \in Q' |\ g^{-1}z \in U_k\}$. Then (Ω'_k) is a finite open cover of Q'. Let Ω_k be an open subspace of $Q_1 \times Q_2$ such that $\Omega'_k = \Omega_k \cap Q'$. Then $\{Q_1 \times Q_2 - Q'\} \cup \{\Omega_k\}$ is an open cover of $Q_1 \times Q_2$. Using Lemma 7.15, there exist finite families of Hausdorff open sets (W_i) and (V_j) which cover Q_1 and Q_2, such that for all i, j and for all $(g, z) \in W_i \times_{G_1^{(0)}} V_j$, there exists k such that $g^{-1}z \in U_k$.
Thus, we can assume by linearity and by Lemmas 4.3 and 7.11 that $\xi \in \mathcal{E}_V^0$, $a \in C_c(W)$, $U = W^{-1}V$, and U, V and W are open and Hausdorff.
Let $\Omega = \{(g, S) \in W^{-1} \times U' |\ g^{-1}q_U(S) \in V\}$. Then the map $(g, S) \mapsto (g^{-1}, g^{-1}S)$ is a homeomorphism from Ω onto $W \times_{r,\rho\circ q_V} V'$. Therefore, the map $(g, z) \mapsto \xi(g^{-1}z)a(g^{-1})$ belongs to $C_c(\Omega) \subset C_c(G_1 \times_{r,\rho\circ q_V} U')$. By Lemma 4.8,

$$S \mapsto (\xi a)(S) = \int_{g\in G_1^{\rho\circ q_V(S)}} \xi(g^{-1}S)a(g^{-1})\, \lambda^{\rho\circ q_V(S)}(\mathrm{d}g)$$

belongs to $C_c(U')$. It is immediate that $(\xi a)(S) = \sqrt{\#S}(\xi a)(q(S))$ for all $S \in U'$, therefore $\xi a \in \mathcal{E}_U^0$. This completes the proof that $\xi a \in \mathcal{E}_Z^0$.

Finally, it is not hard to check that $\langle \xi, \eta a\rangle = \langle \xi, \eta\rangle * a$. Therefore, the completion $\mathcal{E}_Z$ of $\mathcal{E}_Z^0$ with respect to the norm $\|\xi\| = \|\langle \xi, \xi\rangle\|^{1/2}$ is a $C_r^*(G_1)$-Hilbert module.

Let us now construct a morphism $\pi\colon C_r^*(G_2) \to \mathcal{L}(\mathcal{E}_Z)$. For every $\xi \in \mathcal{E}_Z^0$ and every $b \in C_c(G_2)$, define $b\xi$ by Eqn.(4). Let us check that $b\xi \in \mathcal{E}_Z^0$. As above, by linearity we may assume that $\xi \in \mathcal{E}_V^0$, $b \in C_c(W)$ and $VW^{-1} \subset U$, where $V \subset Z$, $U \subset Z$ and $W \subset G_2$ are open and Hausdorff.
Let $\Phi(S, \gamma) = (S\gamma, \gamma)$. Then Φ is a homeomorphism from $\Omega = \{(S, \gamma) \in U' \times_{\sigma\circ q_U, r} W |\ q_U(S)\gamma \in V\}$ onto $V' \times_{\sigma\circ q_V, s} W$. Let $F(z, \gamma) = b(\gamma)\xi(z\gamma)$. Since $F = (\xi \otimes b) \circ \Phi$, F is an element of $C_c(\Omega) \subset C_c(U' \times_{\sigma\circ q_U, r} W)$. By Lemma 4.8, $b\xi \in C_c(U')$.

It is immediate that $(b\xi)(S) = \sqrt{\#S}(b\xi)(q(S))$. Therefore, $b\xi \in \mathcal{E}_U^0 \subset \mathcal{E}_Z^0$.
Let us prove that $\|b\xi\| \le \|b\|\,\|\xi\|$. Let

$$\zeta(\gamma) = \int_{g\in G_1^x} \eta(g)\xi(g^{-1}z\gamma)\,\lambda^x(dg),$$

where $z \in Z$ such that $\rho(z) = r(g)$ is arbitrary. From (5),

$$\langle \eta, \pi_{G_1,x}(\langle \xi,\xi\rangle)\eta\rangle = \|\zeta\|^2_{L^2(G_2^{\sigma(z)})}.$$

A similar calculation shows that

$$\begin{aligned}\langle \eta, \pi_{G_1,x}(\langle b\xi, b\xi\rangle)\eta\rangle &= \int_{\gamma\in G_2^{\sigma(z)}} \lambda^{\sigma(z)}(\mathrm{d}\gamma)\left|\int_{g\in G_1^x} \eta(g)\xi(g^{-1}z\gamma\gamma')b(\gamma')\,\lambda^{s(\gamma)}(\mathrm{d}\gamma')\right|^2\\ &= \langle b\zeta, b\zeta\rangle \le \|b\|^2\|\zeta\|^2.\end{aligned}$$

By density of $C_c(G_2^x)$ in $L^2(G_2^x)$, $\|\pi_{G_1,x}(\langle b\xi, b\xi\rangle)\| \le \|b\|^2\|\pi_{G_1,x}(\langle \xi,\xi\rangle)\|$. Taking the supremum over $x \in G_1^{(0)}$, we get $\|b\xi\| \le \|b\|\,\|\xi\|$. It follows that $b \mapsto (\xi \mapsto b\xi)$ extends to a $*$-morphism $\pi\colon C_r^*(G_2) \to \mathcal{L}(\mathcal{E}_Z)$.

Finally, suppose now that (Z,ρ,σ) is proper, and let us show that $C_r^*(G_2)$ maps to $\mathcal{K}(\mathcal{E}_Z)$.
For every η, $\zeta \in \mathcal{E}_Z^0$, denote by $T_{\eta,\zeta}$ the operator $T_{\eta,\zeta}(\xi) = \eta\langle\zeta,\xi\rangle$. Compact operators are elements of the closed linear span of $T_{\eta,\zeta}$'s. Let us write an explicit formula for $T_{\eta,\zeta}$:

$$\begin{aligned}T_{\eta,\zeta}(\xi)(z) &= \int_{g\in G_1^{\rho(z)}} \eta(g^{-1}z)\langle\zeta,\xi\rangle(g^{-1})\,\lambda^{\rho(z)}(\mathrm{d}g)\\ &= \int_{g\in G_1^{\rho(z)}} \eta(g^{-1}z)\int_{\gamma\in G_2^{\sigma(z)}} \overline{\zeta(g^{-1}z\gamma)}\xi(z\gamma)\,\lambda^{\sigma(z)}(\mathrm{d}\gamma)\lambda^{\rho(z)}(\mathrm{d}g).\end{aligned}$$

Let $b \in C_c(G_2)$, let us show that $\pi(b) \in \mathcal{K}(\mathcal{E}_Z)$. Let K be a quasi-compact subspace of G_2 such that $b^{-1}(\mathbb{C}^*) \subset K$. Since (Z,ρ,σ) is a proper generalized morphism, there exists a quasi-compact subspace Q of Z such that $\sigma^{-1}(r(K)) \subset G_1\mathring{Q}$. Before we proceed, we need a lemma:

LEMMA 7.16. *Let G_2 be a locally compact groupoid acting freely and properly on a locally compact space Z with momentum map $\sigma\colon Z \to G_2^{(0)}$. Then for every $(z_0,\gamma_0) \in Z \rtimes G_2$, there exists a Hausdorff open neighborhood Ω_{z_0,γ_0} of (z_0,γ_0) such that*

- *$U = \{z_1\gamma_1 |\ (z_1,\gamma_1) \in \Omega_{z_0,\gamma_0}\}$ is Hausdorff;*
- *there exists a Hausdorff open neighborhood W of γ_0 such that $\forall\gamma \in G_2$, $\forall z \in pr_1(\Omega_{z_0,\gamma_0})$, $\forall z' \in U$, $z' = z\gamma \implies \gamma \in W$.*

Proof. Let $R = \{(z,z') \in Z \times Z |\ \exists\gamma \in G_2,\ z' = z\gamma\}$. Since the G_2-action is free and proper, there exists a continuous function $\phi\colon R \to G_2$ such that $\phi(z,z\gamma) = \gamma$. Let W be an open Hausdorff neighborhood of γ_0. By continuity of ϕ, there exist open Hausdorff neighborhoods V and U_0 of z_0 and $z_0\gamma_0$ such that for all $(z,z') \in R \cap (V \times U_0)$, $\phi(z,z') \in W$. By continuity of the action,

there exists an open neighborhood Ω_{z_0,γ_0} of (z_0,γ_0) such that $\forall(z_1,\gamma_1)\in\Omega_{z_0,\gamma_0}$, $z_1\gamma_1\in U_0$ and $z_1\in V$. $\square$

By Lemma 7.15, there exist finite covers (V_i) of Q and (W_j) of K such that for every i, j, $(Z\times_{G_2^{(0)}} G_2)\cap(V_i\times W_j)\subset\Omega_{z_0,\gamma_0}$ for some (z_0,γ_0).

By Lemma 6.2 applied to the groupoid $(G_1\ltimes Z)_{V_i}^{V_i}$, for all i there exists $c_i'\in C_c(V_i')_+$ such that $c_i'(S)=(\#S)c_i'(q_{V_i}(S))$ for all $S\in V_i'$, and such that $\sum_i c_i'\geq 1$ on Q. Let

$$f_i(z)=\int_{g\in G_1^{\rho(z)}} c_i'(g^{-1}z)\,\lambda^{\rho(z)}(\mathrm{d}g)$$

and let $f=\sum_i f_i$. As in the proof of Theorem 6.3, one can show that for every Hausdorff open subspace V of Z and every $h\in C_c(V)$, $(g,z)\mapsto h(z)c_i'(g^{-1}z)$ belongs to $C_c(G\ltimes Z)$, therefore hf_i is continuous on V. Since h is arbitrary, it follows that f_i is continuous, thus f is continuous. Moreover, f is G_1-equivariant, nonnegative, and $\inf_Q f>0$. Therefore, there exists $f_1\in C_c(G_1\backslash Z)$ such that $f_1(z)=1/f(z)$ for all $z\in Q$. Let $c_i(z)=f_1(z)c_i'(z)$. Let

$$T_i(\xi)(z)=\int_{g\in G_1^{\rho(z)}}\int_{\gamma\in G_2^{\sigma(z)}} c_i(g^{-1}z)b(\gamma)\xi(z\gamma)\,\lambda^{\rho(z)}(\mathrm{d}g)\lambda^{\sigma(z)}(\mathrm{d}\gamma).$$

Then $\pi(b)=\sum_i T_i$, therefore it suffices to show that T_i is a compact operator for all i.

By linearity and by Lemma 4.3, one may assume that $b\in C_c(W_j)$ for some j. Then, by construction of V_i (see Lemma 7.16), there exist open Hausdorff sets $U\subset Z$ and $W\subset G_2$ such that $\{\gamma\in G_2|\ \exists(z,z')\in V_i\times U,\ z'=z\gamma\}\subset W$, and $\{z\gamma|\ (z,\gamma)\in V_i\times_{\sigma,r}W\}\subset U$.

The map $(z,z\gamma)\mapsto c(z)b(\gamma)$ defines an element of $C_c(V_i'\times U)$. Let $L_1\times L_2\subset V_i\times U$ compact such that $(z,z\gamma)\mapsto c(z)b(\gamma)$ is supported on $q_{V_i}^{-1}(L_1)\times L_2$. By Lemma 6.2 applied to the groupoids $(G_1\ltimes Z)_{V_i}^{V_i}$ and $(G_1\ltimes Z)_U^U$, there exist $d_1\in C_c(V_i')_+$ and $d_2\in C_c(U')_+$ such that $d_1>0$ on L_1 and $d_2>0$ on L_2, $d_1(S)=\sqrt{\#S}d_1(q_{V_i}(S))$ for all $S\in V_i'$, and $d_2(S)=\sqrt{\#S}d_2(q_U(S))$ for all $S\in U'$. Let

$$f(z,z\gamma)=\frac{c(z)b(\gamma)}{d_1(z)d_2(z\gamma)}.$$

Then $f\in C_c(V_i\times_{G_1^{(0)}}U)$. Therefore, f is the uniform limit of a sequence $f_n=\sum\alpha_{n,k}\otimes\overline{\beta_{n,k}}$ in $C_c(V_i)\otimes C_c(U)$ such that all the f_n are supported in a fixed compact set. Then T_i is the norm-limit of $\sum_k T_{d_1\alpha_{n,k},d_2\beta_{n,k}}$, therefore it is compact.

REMARK 7.17. *The construction in Theorem 7.8 is functorial with respect to the composition of generalized morphisms and of correspondences. We don't include a proof of this fact, as it is tedious but elementary. It is an easy exercise when G_1 and G_2 are Hausdorff.*

REFERENCES

[1] Abels, H. Parallelizability of proper actions, global K-slices and maximal compact subgroups. *Math. Ann.* 212 (1974/75), 1–19.

[2] Bourbaki, N.: *Topologie Générale.* Masson, Paris (1990).

[3] Chabert, J.; Echterhoff, S.; Meyer, R. Deux remarques sur l'application de Baum-Connes. *C. R. Acad. Sci. Paris Sér. I Math.* 332 (2001), no. 7, 607–610.

[4] Haefliger, A. *Groupoïdes d'holonomie et classifiants*, Astérisque 116 (1984), 70–97.

[5] Hilsum, M. and Skandalis, G.: Morphismes K-orientés d'espaces de feuilles et fonctorialité en théorie de Kasparov (d'après une conjecture d'A. Connes). *Ann. Sci. École Norm. Sup.* (4) 20(1987), no. 3, 325–390.

[6] Khoshkam, M.; Skandalis, G. Regular representation of groupoid C^*-algebras and applications to inverse semigroups. *J. Reine Angew. Math.* 546 (2002), 47–72.

[7] Landsman, N. P. Operator algebras and Poisson manifolds associated to groupoids. *Comm. Math. Phys.* 222 (2001), no. 1, 97–116.

[8] Landsman, N. P. Quantized reduction as a tensor product. Quantization of singular symplectic quotients, 137–180, *Progr. Math.*, 198, Birkhäuser, Basel, 2001.

[9] Le Gall, P.Y.: Théorie de Kasparov équivariante et groupoïdes. I. *K-Theory* 16(1999), no. 4, 361–390.

[10] Le Gall, P.Y.; Monthubert, B. K-theory of the indicial algebra of a manifold with corners. *K-Theory* 23 (2001), no. 2, 105–113.

[11] Macho-Stadler, M. and O'Uchi, M.: Correspondence of groupoid C^*-algebras, *J. Operator Theory* 42(1999), 103-119.

[12] Moerdijk, I. *Toposes and groupoids*, in *Categorical algebra and its applications*, 280–298, LNM 1348, Springer, Berlin, 1988.

[13] Mrčun, J., Functoriality of the bimodule associated to a Hilsum-Skandalis map, *K-Theory* 18 (1999), 235–253.

[14] Muhly, P., Renault, J. and Williams, D.: Equivalence and isomorphism for groupoid C^*-algebras, *J. Operator Theory* 17(1987), 3–22.

[15] Paterson, A: *Groupoids, inverse semigroups, and their operator algebras.* Progress in Mathematics, 170. Birkhäuser Boston, Inc., Boston, MA, 1999.

[16] Renault, J: *A groupoid approach to C^*-algebras*, Springer, 1980, LNM no. 793.

[17] Stachura, P. C^*-algebra of a differential groupoid. With an appendix by S. Zakrzewski. Banach Center Publ., 51, Poisson geometry (Warsaw, 1998), 263–281, Polish Acad. Sci., Warsaw, 2000.

Jean-Louis Tu
University Paris VI
Institut de Mathématiques
175, rue du Chevaleret
75013 Paris, France.
tu@math.jussieu.fr